Basic Concepts of Fuzzy Logic

- Apparatus of fuzzy logic is built on:

  ⇒ Fuzzy sets: describe the value of variables

  Possibility distributions: constraints on the value of a variable

  Linguistic variables: qualitatively and quantitatively described by fuzzy sets

  Fuzzy if-then rules: knowledge

*Fuzzy Logic: Intelligence, Control, and Information - J. Yen and R. Langari, Prentice Hall 1999

Fuzzy sets

A fuzzy set is a set with a smooth boundary.

An element of a fuzzy set can belong to that set partially to a degree and the set does not have crisp boundaries.

A fuzzy set is defined by a functions that maps objects in a domain of concern into their membership value in a set.

Such a function is called the *membership function*.

Fuzzy sets

Definition: let X be a non-empty set and be called the universe of discourse. A fuzzy set A⊂X is characterized by the membership function

$$\mu_A : X \rightarrow [0,1]$$

where $\mu_A(x)$ is a grade (degree) of membership of x in set A.

Fuzzy sets

$$\mu_A : X \rightarrow [0,1]$$

Note : since {0,1}∈ [0,1]
all crisp sets are fuzzy sets!

Fuzzy sets

Definition of fuzzy sets:

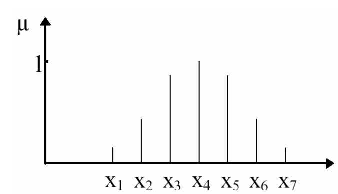
Fuzzy set A can be represented as a set of ordered pairs

$$A = \{(x, \mu_A(x)) | x \in X\}$$

Fuzzy sets

Discrete example:

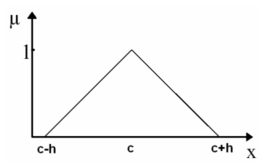
μ_A = 0.1/x1 + 0.4/x2 +0.8/x3 + 1.0/x4 + 0.8/x5 + 0.4/x6 + 0.1/x7



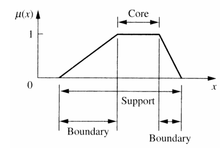
Fuzzy sets

Continuous example:

$$\mu_A(x) = \begin{cases} 1 + \dfrac{x-c}{h}, & x \in [c-h, c] \\ 1 - \dfrac{x-c}{h}, & x \in [c, c+h] \\ 0, & \text{otherwise} \end{cases}$$



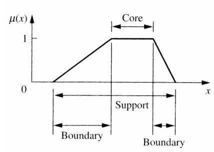
Properties of fuzzy sets



Support : support of a fuzzy set A is a crisp set that contains all elements of A with non-zero membership grade:

$$\text{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}$$

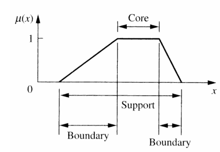
Properties of fuzzy sets



Core: comprises those elements *x* of the universe such that μ_A (x) = 1.

$$\text{core}(A) = \{x \in X \mid \mu_A(x) = 1\}$$

Properties of fuzzy sets



Boundary : boundaries comprise those elements *x* of the universe such that 0< μ_A (x) <1

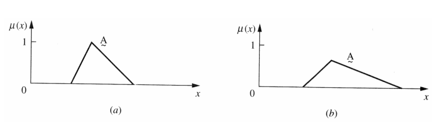
$$\text{bnd}(A) = \{x \in X \mid 0 < \mu_A(x) < 1\}$$

Properties of fuzzy sets

Height : the height of a fuzzy set A id defined as

$$\text{hgt}(A) = \sup_{x \in X} \mu_A(x)$$

Set A is called normal if hgt(A)=1 and subnormal if hgt(A)<1



Fuzzy sets that are normal (*a*) and subnormal (*b*).

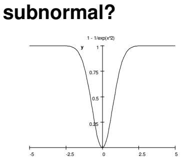
Properties of fuzzy sets

Question?

Is the fuzzy set defined as

$$\mu_A(x) = 1 - 1/e^{-x^2}$$

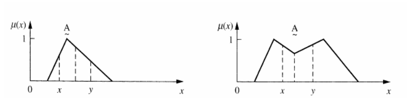
Normal or subnormal?



Properties of fuzzy sets

Convex Fuzzy set: a fuzzy set A is convex iff

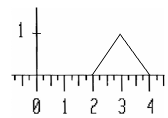
$$\forall x, y \in X \quad \text{and} \quad \forall \lambda \in [0,1]$$
$$\mu_A(\lambda x + (1-\lambda) y) \geq \min(\mu_A(x), \mu_A(y))$$



Convex, normal fuzzy set — Non convex, normal fuzzy set

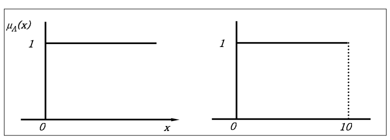
Properties of fuzzy sets

Fuzzy number: a fuzzy set A is a fuzzy number if the fuzzy set is
- **Convex**
- **Normal**
- **The core consists of one value only**
- **MF is piecewise continuous**

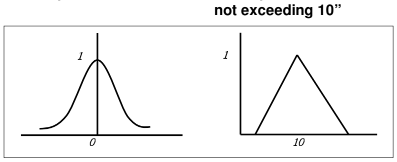


Example: fuzzy 3

Properties of fuzzy sets



Set "positive number" — **Set "positive number not exceeding 10"**



Set "number near 0" — **Set "number near 10"**

Operations on fuzzy sets

1. **Empty set**

$$\mu_{\varnothing} \equiv 0$$

2. **Basic set (universe)**

$$\mu_X \equiv 1$$

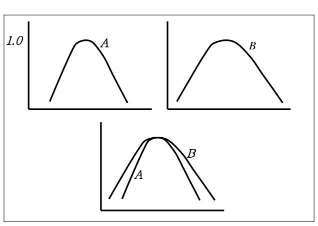
3. **Identity**

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \quad \forall x \in X$$

Operations on fuzzy sets

4. **Subset**

$$A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x) \quad \forall x \in X$$



Operations on fuzzy sets

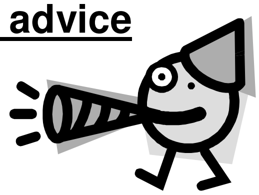
5. **Union**

Axioms for union function

$U : [0,1] \times [0,1] \rightarrow [0,1] \qquad \mu_{A \cup B}(x) = U[\mu_A(x), \mu_B(x)]$

- $U(0,0) = 0$, $U(0,1) = 1$, $U(1,0) = 1$, $U(1,1) = 1$
- $U(a,b) = U(b,a)$ (Commutativity)
- If $a \leq a'$ and $b \leq b'$, $U(a, b) \leq U(a', b')$ (monotonicity).
- $U(U(a, b), c) = U(a, U(b, c))$ (Associativity)
- Function U is continuous.
- $U(a, a) = a$ (idempotency)

Fuzzy advice



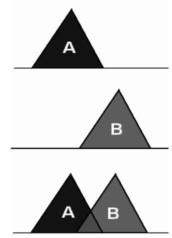
Do not sleep in class!!!

For sleep-deprived: *SL119:Linear Sleep*
Advanced class in mattress manipulation.
Topics include unconsciousness and hibernation.

Operations on fuzzy sets

5. **Union**

$$U[\mu_A, \mu_B] = \max[\mu_A, \mu_B],$$
$$\forall x \in X : \mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$



Operations on fuzzy sets

6. Intersection

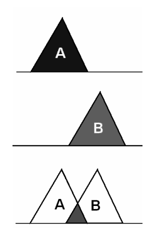
Axioms for intersection function
I:[0,1] × [0,1] → [0,1] $\mu_{A \cap B}(x) = I[\mu_A(x), \mu_B(x)]$

- $I(1, 1) = 1, I(1, 0) = 0, I(0, 1) = 0, I(0, 0) = 0$
- $I(a, b) = I(b, a)$, Commutativity.
- If $a \le a'$ and $b \le b'$, $I(a, b) \le I(a', b')$, monotonicity.
- $I(I(a, b), c) = I(a, I(b, c))$, Associativity.
- I is a continuous function.
- $I(a, a) = a$, idempotency.

Operations on fuzzy sets

6. **Intersection**

$$I[\mu_A, \mu_B] = \min[\mu_A, \mu_B]$$
$$\forall x \in X : \mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$



Operations on fuzzy sets

7. **Complement**

Axioms for Complement function
C: [0,1] → [0,1]

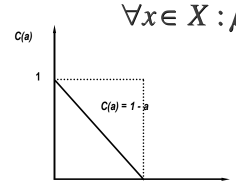
- Boundary conditions $C(0) = 1, C(1) = 0$
- C is monotonic non-increasing
  $a,b \in [0,1]$ if $a < b$, then $C(a) \ge C(b)$
- C is a continuous function.
- C is involutive $C(C(a)) = a$ for all $a \in [0,1]$

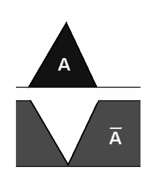
Operations on fuzzy sets

7. **Complement**

Standard complement function:

$$C(a) = 1 - a$$
$$\forall x \in X : \mu_{\bar{A}}(x) = 1 - \mu_A(x)$$





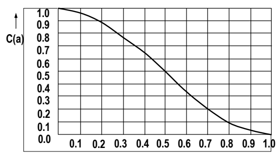
Operations on fuzzy sets

7. **Complement**

Continuous complement function:

$$C(a) = 1/2(1+\cos(\pi a))$$

$$\forall x \in X : \mu_{\bar{A}}(x) = 1/2(1+\cos(\pi\mu_A(x)))$$



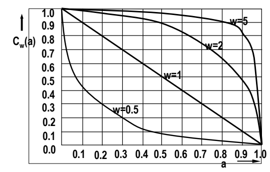
Operations on fuzzy sets

7. **Complement**

Yager complement function:

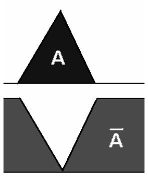
$$C_w(a) = (1-a^w)^{1/w}, \qquad w \in (-1, \infty)$$

$$\forall x \in X : \mu_{\bar{A}}(x) = (1-\mu_A(x)^w)^{1/w}$$



Operations on fuzzy sets

7. **Complement**



Note: the laws of excluded middle

$$A \cap \bar{A} = \varnothing$$

and the law of contradiction

$$A \cup \bar{A} = X$$

Are not valid for fuzzy sets!

Operations on fuzzy sets

Properties of fuzzy operations:

(1) Involution	$\bar{\bar{A}} = A$
(2) Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
(3) Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
(4) Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(5) Idempotency	$A \cup A = A$
	$A \cap A = A$
(6) Absorption	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$

Operations on fuzzy sets

Properties of fuzzy operations:

(7) Absorption by X and $\varnothing$	$A \cup X = X$
	$A \cap \varnothing = \varnothing$
(8) Identity	$A \cup \varnothing = A$
	$A \cap X = A$
(9) De Morgan's law	$\overline{A \cap B} = \bar{A} \cup \bar{B}$
	$\overline{A \cup B} = \bar{A} \cap \bar{B}$

(10) Equivalence formula

$$(\bar{A} \cup B) \cap (A \cup \bar{B}) = (\bar{A} \cap \bar{B}) \cup (A \cap B)$$

(11) Symmetrical difference formula

$$(\bar{A} \cap B) \cup (A \cap \bar{B}) = (\bar{A} \cup \bar{B}) \cap (A \cup B)$$

Operations on fuzzy sets

Power of fuzzy sets:

$$\mu_{A^2}(x) = [\mu_A(x)]^2, \quad \forall x \in X$$

$$\mu_{A^m}(x) = [\mu_A(x)]^m, \quad \forall x \in X^m$$

Operations on fuzzy sets

α-cut set

– $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$, α is an arbitrary real number in [0,1]

– α-cut set is a crisp set