FUZZY RELATIONS and COMPOSITION OF FUZZY RELATIONS

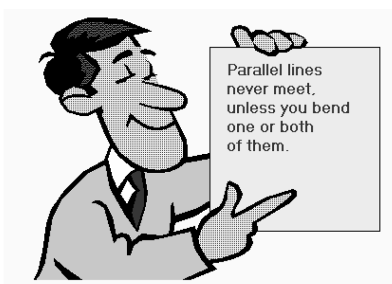
Fuzzy Relations

Fuzzy relation generalizes classical relation into one that allows partial membership and describes a relationship that holds between two or more objects.

Example: a fuzzy relation “Friend” describe the degree of friendship between two persons (in contrast to either being friend or not being friend in classical relation!)

Fuzzy Logic:Intelligence, Control, and Information, J. Yen and R. Langari, PrenticeHall

Fuzzy Advice



Beware of the math!

Crisp Cartesian product

Lets consider properties of crisp relations first and then extend the mechanism to fuzzy sets.

Definition of (crisp) Product set: Let A and B be two non-empty sets, the product set or Cartesian product $A \times B$ is defined as follows,

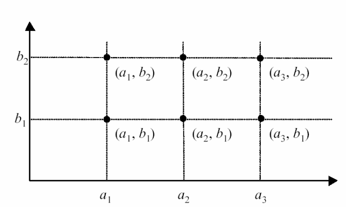
$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

(a set of ordered pairs a,b)

Crisp Cartesian product

Example: When $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$ the Cartesian product yields

$$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$$



Product set $A \times B$

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Crisp Binary Relation

Cartesian product of *n* sets

$$A_1 \times A_2 \times \ldots \times A_n = \prod_{i=1}^{n} A_i = \{(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}$$

Definition of Binary Relation

If A and B are two sets and there is a specific property between elements x of A and y of B, this property can be described using the ordered pair (x, y). A set of such (x, y) pairs, $x \in A$ and $y \in B$, is called a relation R.

$$R = \{ (x,y) \mid x \in A, y \in B \}$$

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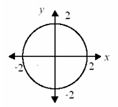
Crisp Binary Relation

Examples of Binary relations:

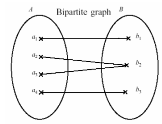
Coordinate diagram



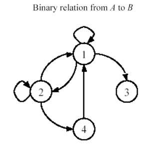
Relation of $x^2 + y^2 = 4$

Relation matrix

R	b_1	b_2	b_3
a_1	1	0	0
a_2	0	1	0
a_3	0	1	0
a_4	0	0	1



Binary relation from A to B



Directed graph

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Crisp n-ary Relation

Definition of *n*-ary relation

For sets A_1, A_2, A_3, ..., A_n, the relation among elements $x_1 \in A_1$, $x_2 \in A_2$, $x_3 \in A_3$, ..., $x_n \in A_n$ can be described by n-tuple (x_1, x_2, ..., x_n). A collection of such n-tuples (x_1, x_2, x_3, ..., x_n) is a relation R among A_1, A_2, A_3, ..., A_n.

$$(x_1, x_2, x_3, \dots, x_n) \in R,$$
$$R \subseteq A_1 \times A_2 \times A_3 \times \dots \times A_n$$

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Fuzzy Cartesian Product

Let $\mu_{A_1}(x),\ \mu_{A_2}(x),\ \dots,\ \mu_{A_n}(x)$ be membership functions of $A_1, A_2, \dots, A_n$ for $\forall x_1 \in A_1,\ x_2 \in A_2,\ \dots, x_n \in A_n$.

Then the Cartesian product (the probability for n-tuple (x_1,x_2,…,x_n) to be involved in fuzzy set $A_1 \times A_2 \times \dots \times A_n$ is,

$$\mu_{A_1 \times A_2 \times \dots \times A_n}(x_1, x_2, \dots, x_n) = \min[\mu_{A_1}(x_1), \dots, \mu_{A_n}(x_n)]$$

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Fuzzy Cartesian Product: Example

Let A be a fuzzy set defined on a universe of three discrete temperatures, X = {x_1,x_2,x_3}, and B be a fuzzy set defined on a universe of two discrete pressures, Y = {y_1,y_2}
Fuzzy set A represents the "ambient" temperature and fuzzy set B the "near optimum" pressure for a certain heat exchanger, and the Cartesian product might represent the conditions (temperature-pressure pairs) of the exchanger that are associated with "efficient" operations. For example, let

$$\left.\begin{aligned} A &= \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \\ B &= \frac{0.3}{y_1} + \frac{0.9}{y_2} \end{aligned}\right\} \qquad A \times B = R = \begin{matrix} & y_1 & y_2 \\ x_1 & 0.2 & 0.2 \\ x_2 & 0.3 & 0.5 \\ x_3 & 0.3 & 0.9 \end{matrix}$$

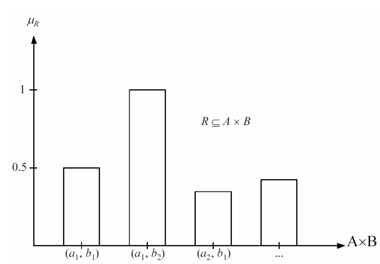
Fuzzy Logic with Engineering Applications: Timothy J. Ross, McGraw-Hill

Fuzzy Relation

A fuzzy relation R is a mapping from the Cartesian space X x Y to the interval [0,1], where the strength of the mapping is expressed by the membership function of the relation $\mu_R(x,y)$

$$\mu_R : A \times B \to [0, 1]$$
$$R = \{((x, y), \mu_R(x, y)) \mid \mu_R(x, y) \geq 0,\ x \in A,\ y \in B\}$$

Fuzzy Relation



Fuzzy relation as a fuzzy set

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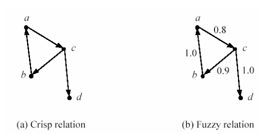
Fuzzy Relation

Crisp relation vs. Fuzzy relation



Corresponding fuzzy relation matrix

A \ A	a	b	c	d
a	0.0	0.0	0.8	0.0
b	1.0	0.0	0.0	0.0
c	0.0	0.9	0.0	1.0
d	0.0	0.0	0.0	0.0

Composition of Fuzzy Relations

Two fuzzy relations R and S are defined on sets A, B and C. That is, $R \subseteq A \times B$, $S \subseteq B \times C$. The composition $S \bullet R = SR$ of two relations R and S is expressed by the relation from A to C:

For $(x, y) \in A \times B$, $(y, z) \in B \times C$,

$$\mu_{S \bullet R}(x, z) = \max_y [\min(\mu_R(x, y), \mu_S(y, z))]$$

$$= \vee_y [\mu_R(x, y) \wedge \mu_S(y, z)]$$

$M_{S \bullet R} = M_R \bullet M_S$ (matrix notation)

(max-min composition)

Composition of Fuzzy Relations

Example: Consider fuzzy relations $R \subseteq A \times B$, $S \subseteq B \times C$. The sets A, B and C shall be the sets of events. By the relation R, we can see the possibility of occurrence of B after A, and by S, that of C after B. For example, by M_R, the possibility of $a \in B$ after $1 \in A$ is 0.1. By M_S, the possibility of occurrence of α after a is 0.9.

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

$\mu_R(x, y)$

$\mu_S(y, z))$

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

0.1 0.2 0.0 1.0
0.9 0.2 0.8 0.4
min 0.1 0.2 0.0 0.4

max=0.4

Here, we can not guess the possibility of C when A is occurred. So our main job now will be the obtaining the composition $S \bullet R \subseteq A \times C$. The following matrix $M_{S \bullet R}$ represents this composition

$S \bullet R$	α	β	γ
1	0.4	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

Composition of Fuzzy Relations

Example:

$X = \{x_1, x_2\}, Y = \{y_1, y_2\},$ and $Z = \{z_1, z_2, z_3\}$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{matrix} & y_1 & y_2 \\ x_1 & 0.7 & 0.5 \\ x_2 & 0.8 & 0.4 \end{matrix} \quad \text{and} \quad \tilde{S} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.9 & 0.6 & 0.5 \\ y_2 & 0.1 & 0.7 & 0.5 \end{matrix}$$

Using max-min composition,

$$\left.\begin{aligned} \mu_{\tilde{T}}(x_1, z_1) &= \vee_{y \in Y}(\mu_{\tilde{R}}(x_1, y) \wedge \mu_{\tilde{S}}(y, z_1)) \\ &= \max[\min(0.7, 0.9), \min(0.5, 0.1)] \\ &= 0.7 \end{aligned}\right\} \quad \tilde{T} = \begin{matrix} & z_1 & z_2 & z_3 \\ x_1 & 0.7 & 0.6 & 0.5 \\ x_2 & 0.8 & 0.6 & 0.4 \end{matrix}$$

Composition of Fuzzy Relations

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For $(x, y) \in A \times B$, $(y, z) \in B \times C$,

$$\mu_{S \bullet R}(x, z) = \max_y [\mu_R(x, y) \bullet \mu_S(y, z)]$$

$$= \vee_y [\mu_R(x, y) \bullet \mu_S(y, z)]$$

$M_{S \bullet R} = M_R \bullet M_S$ (matrix notation)

(max-product composition)

Composition of Fuzzy Relations

Max-product example:

$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\},$ and $Z = \{z_1, z_2, z_3\}$

Consider the following fuzzy relations:

$$\tilde{R} = \begin{matrix} & y_1 & y_2 \\ x_1 & 0.7 & 0.5 \\ x_2 & 0.8 & 0.4 \end{matrix} \quad \text{and} \quad \tilde{S} = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.9 & 0.6 & 0.5 \\ y_2 & 0.1 & 0.7 & 0.5 \end{matrix}$$

Using max-product composition,

$$\left.\begin{aligned} \mu_{\tilde{T}}(x_2, z_2) &= \vee_{y \in Y}(\mu_{\tilde{R}}(x_2, y) \bullet \mu_{\tilde{S}}(y, z_2)) \\ &= \max[(0.8, 0.6), (0.4, 0.7)] \\ &= 0.48 \end{aligned}\right\} \quad \tilde{T} = \begin{matrix} & z_1 & z_2 & z_3 \\ x_1 & .63 & .42 & .25 \\ x_2 & .72 & .48 & .20 \end{matrix}$$

Application: Computer Engineering

Problem: In computer engineering, different logic families are often compared on the basis of their power-delay product. Consider the fuzzy set F of logic families, the fuzzy set D of delay times (ns), and the fuzzy set P of power dissipations (mw).

If F = {NMOS,CMOS,TTL,ECL,JJ}, D = {0.1,1,10,100},

P = {0.01,0.1,1,10,100}

Suppose R_1 = D x F and R_2 = F x P

$$\tilde{R}_1 = \begin{array}{c} \\ 0.1 \\ 1 \\ 10 \\ 100 \end{array}\begin{array}{c} \begin{array}{ccccc} N & C & T & E & J \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & .6 & 1 \\ 0 & .1 & .5 & 1 & 0 \\ .4 & 1 & 1 & 0 & 0 \\ 1 & .2 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{R}_2 = \begin{array}{c} \\ N \\ C \\ T \\ E \\ J \end{array}\begin{array}{c} \begin{array}{ccccc} .01 & .1 & 1 & 10 & 100 \end{array} \\ \begin{bmatrix} 0 & .4 & 1 & .3 & 0 \\ .2 & 1 & 0 & 0 & 0 \\ 0 & 0 & .7 & 1 & 0 \\ 0 & 0 & 0 & 1 & .5 \\ 1 & .1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Application: Computer Engineering (Cont)

We can use max-min composition to obtain a relation between delay times and power dissipation: i.e., we can compute $R_3 = R_1 \circ R_2$ or $\mu_{R_3} = \vee(\mu_{R_1} \wedge \mu_{R_2})$

$$R_3 = \begin{array}{c} \\ 0.1 \\ 1 \\ 10 \\ 100 \end{array}\begin{array}{c} \begin{array}{ccccc} .01 & .1 & 1 & 10 & 100 \end{array} \\ \begin{bmatrix} 1 & .1 & 0 & .6 & .5 \\ .1 & .1 & .5 & 1 & .5 \\ .2 & 1 & .7 & 1 & 0 \\ .2 & .4 & 1 & .3 & 0 \end{bmatrix} \end{array}$$

Application: Fuzzy Relation Petite

Fuzzy Relation Petite defines the degree by which a person with a specific height and weight is considered petite. Suppose the range of the height and the weight of interest to us are {5', 5'1", 5'2", 5'3", 5'4",5'5",5'6"}, and {90, 95,100, 105, 110, 115, 120, 125} (in lb). We can express the fuzzy relation in a matrix form as shown below:

$P =$

	90	95	100	105	110	115	120	125
5'	1	1	1	1	1	1	.5	.2
5'1"	1	1	1	1	1	.9	.3	.1
5'2"	1	1	1	1	1	.7	.1	0
5'3"	1	1	1	1	.5	.3	0	0
5'4"	.8	.6	.4	.2	0	0	0	0
5'5"	.6	.4	.2	0	0	0	0	0
5'6"	0	0	0	0	0	0	0	0

Application: Fuzzy Relation Petite

Once we define the petite fuzzy relation, we can answer two kinds of questions:

- What is the degree that a female with a specific height and a specific weight is considered to be petite?
- What is the possibility that a petite person has a specific pair of height and weight measures?

$P =$

	90	95	100	105	110	115	120	125
5'	1	1	1	1	1	1	.5	.2
5'1"	1	1	1	1	1	.9	.3	.1
5'2"	1	1	1	1	1	.7	.1	0
5'3"	1	1	1	1	.5	.3	0	0
5'4"	.8	.6	.4	.2	0	0	0	0
5'5"	.6	.4	.2	0	0	0	0	0
5'6"	0	0	0	0	0	0	0	0

Application: Fuzzy Relation Petite

Given a two-dimensional fuzzy relation and the possible values of one variable, infer the possible values of the other variable using similar fuzzy composition as described earlier.

Definition: Let X and Y be the universes of discourse for variables x and y, respectively, and x_i and y_j be elements of X and Y. Let R be a fuzzy relation that maps X×Y to [0,1] and the possibility distribution of X is known to be $P_x(x_i)$. The compositional rule of inference infers the possibility distribution of Y as follows:

max-min composition: $\Pi_Y(y_j) = \max_{x_i}(\min(\Pi_X(x_i), \Pi_R(x_i, y_j)))$

max-product composition: $\Pi_Y(y_j) = \max_{x_i}(\Pi_X(x_i) \times \Pi_R(x_i, y_j))$

Application: Fuzzy Relation Petite

Problem: We may wish to know the possible weight of a petite female who is about 5'4".
Assume About 5'4" is defined as

About-5'4" = {0/5', 0/5'1", 0.4/5'2", 0.8/5'3", 1/5'4", 0.8/5'5", 0.4/5'6"}

Using max-min compositional, we can find the weight possibility distribution of a petite person about 5'4" tall:

$\tilde{P} =$

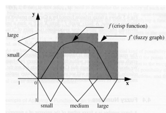
	90	95	100	105	110	115	120	125
5'	1	1	1	1	1	1	.5	.2
5'1"	1	1	1	1	1	.9	.3	.1
5'2"	1	1	1	1	1	.7	.1	0
5'3"	1	1	1	1	.5	.3	0	0
5'4"	.8	.6	.4	.2	0	0	0	0
5'5"	.6	.4	.2	0	0	0	0	0
5'6"	0	0	0	0	0	0	0	0

$$\Pi_{weight}(90) = (0 \wedge 1) \vee (0 \wedge 1) \vee (.4 \wedge 1) \vee (.8 \wedge 1) \vee (1 \wedge .8) \vee (.8 \wedge .6) \vee (.4 \wedge 0) = 0.8$$

Similarly, we can compute the possibility degree for other weights. The final result is

$$\Pi_{weight} = \{0.8/90, 0.8/95, 0.8/100, 0.8/105, 0.5/110, 0.4/115, 0.1/120, 0/125\}$$

Fuzzy Graphs



- A fuzzy relation may not have a meaningful linguistic label.
- Most fuzzy relations used in real-world applications do not represent a concept, rather they represent a functional mapping from a set of input variables to one or more output variables.
- Fuzzy rules can be used to describe a fuzzy relation from the observed state variables to a control decision (using fuzzy graphs)
- A fuzzy graph describes a functional mapping between a set of input linguistic variables and an output linguistic variable.