FUZZY LOGIC

Logic

logic

\Log"ic\, n. 1. The science or art of exact reasoning, or of pure and formal thought, or of the laws according to which the processes of pure thinking should be conducted; the science of the formation and application of general notions; the science of generalization, judgment, classification, reasoning, and systematic arrangement; correct reasoning.

Source: *Webster's Revised Unabridged Dictionary, © 1996, 1998 MICRA, Inc.*

logic

n 1: the branch of philosophy that analyzes inference 2: reasoned and reasonable judgment; "it made a certain kind of logic" 3: the principles that guide reasoning within a given field or situation; "economic logic requires it"; "by the logic of war" 4: a system of reasoning [syn: logical system, system of logic]

Source: *WordNet ® 1.6, © 1997 Princeton University*

Inference

inference

\In"fer*ence\, n. [From Infer.]

1. The act or process of inferring by deduction or induction.

2. That which inferred; a truth or proposition drawn from another which is admitted or supposed to be true; a conclusion; a deduction. --Milton.

Usage: Inference, Conclusion. An inference is literally that which is brought in; and hence, a deduction or induction from premises, --something which follows as certainly or probably true. ``An inference is a proposition which is perceived to be true, because of its connection with some known fact.'' ``When something is simply affirmed to be true, it is called a proposition; after it has been found to be true by several reasons or arguments, it is called a conclusion.'' --I. Taylor.

Source: *Webster's Revised Unabridged Dictionary, © 1996, 1998 MICRA, Inc.*

Inference

inference

n : the reasoning involved in making a logical judgment on the basis of circumstantial evidence and prior conclusions rather than on the basis of direct observation [syn: illation]

Source: *WordNet ® 1.6, © 1997 Princeton University*

inference

<logic> The logical process by which new facts are derived from known facts by the application of inference rules.

Source: *The Free On-line Dictionary of Computing, © 1993-2003 Denis Howe*

Classical Proposition logic

As in our ordinary informal language, "**sentence**" is used in the logic. Especially, a sentence having only "true (1)" or "false (0)" as its truth value is called "**proposition**".

- 2 + 4 = 7 (false)
- For every x, if f(x) = sin x, then f′(x) = cos x. (true)
- It rains now. (true or false depending whether it rains or not)

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Classical Proposition logic

The followings are not propositions

- Why are you coming to class?
- He hits 5 home runs in one season.
- x + 5 = 0
- x + y = z

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Classical Proposition Logic

- Logic variable
  - If we represent a proposition as a variable, the variable can have the value true or false.
  - This type of variable is called as a “proposition variable” or “logic variable”

Classical Proposition Logic

- Connectives - combine prepositional variables
  - Negation $\overline{a}$
  - Conjunction $a \wedge b$
  - Disjunction $a \vee b$
  - Implication $a \rightarrow b$
  - etc.

p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	T	F	T
F	F	T	F	F	T

Logic functions

- Logic function
  - a combination of propositional variables by using connectives
- Logic formula
  - Truth values 0 and 1 are logic formulas
  - If v is a logic variable, v and v’ are a logic formulas
  - If a and b represent a logic formulas, $a \wedge b$ and $a \vee b$ are also logic formulas

Tautology and inference rule

- Tautology
  - A “tautology” is a logic formula whose value is always true regardless of its logic variables.
  - A “contradiction” is one which is always false.

Tautology $\overline{(a \rightarrow b)} \rightarrow \overline{b}$

a	b	$a \rightarrow b$	$\overline{(a \rightarrow b)}$	$\overline{b}$	$\overline{(a \rightarrow b)} \rightarrow \overline{b}$
1	1	1	0	0	1
1	0	0	1	1	1
0	1	1	0	0	1
0	0	1	0	1	1

Tautology and inference rule

$(a \wedge (a \rightarrow b)) \rightarrow b$

a	b	$(a \rightarrow b)$	$(a \wedge (a \rightarrow b))$	$(a \wedge (a \rightarrow b)) \rightarrow b$
1	1	1	1	1
1	0	0	0	1
0	1	1	0	1
0	0	1	0	1

This tautology means that
“If a is true and $(a \rightarrow b)$ is true, then b is true.”
or “If a exists and the relation $(a \rightarrow b)$ is true, then b exists.”

Tautology and inference rule

Important inference rules that use tautologies:

- Modus Ponens

$$(a \wedge (a \rightarrow b)) \rightarrow b$$

- Modus Tollens

$$(\sim b \wedge (a \rightarrow b)) \rightarrow \sim a$$

- Hypothetical syllogism

$$(((a \rightarrow b) \wedge (b \rightarrow c)) \rightarrow (a \rightarrow c)$$

Predicate logic

- Predicate logic

  "Predicate logic" is a logic which represents a proposition with the predicate and an individual (object)
  - "Socrates is a man"
    - "Socrates" – object
    - "is a man" – predicate
  - "Two is less than four"
    - "Two", "four" – objects
    - "is less than" - predicate

Predicate logic

Objects in predicate logic can represented by variables. Then a predicate proposition can be evaluated for truth if an element of a universal set is instantiated to the variable.

"x is a man"

- x="Tom", the proposition becomes "Tom is a man"

"x satisfies P" can be denoted P(x)

- is_a_man(Tom)

Quantifiers

- Universal quantifier
  - "for all"
  - Denoted symbolically by $\forall$
- Existential quantifier
  - "there exists"
  - Denoted symbolically by $\exists$

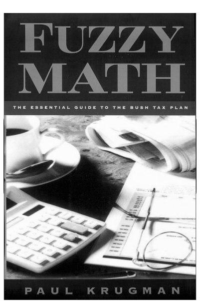
Fuzzy expressions

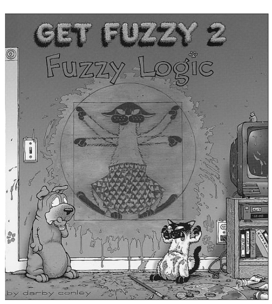
- In a fuzzy expression (formula), a fuzzy proposition can have its truth value in the interval [0,1]

$$f : [0,1] \rightarrow [0,1]$$

- Generalization to n dimensions

$$f : [0,1]^n \rightarrow [0,1]$$





A Fuzzy Proposition:

Not every book on fuzzy logic is a book on fuzzy logic!

Fuzzy Logic

- Definition of Fuzzy logic

  Fuzzy logic is a logic represented by a fuzzy expression (formula) which satisfies the following:
  - Truth values, 0 and 1, and variables x_i ($\in [0,1]$, i = 1, 2, …, n) are fuzzy expressions
  - If f is a fuzzy expression, ~f (not f) is also a fuzzy expression
  - If f and g are fuzzy expressions, $f \wedge g$ and $f \vee g$ are also fuzzy expressions

Operators in fuzzy expressions

- Operators in fuzzy expression
  - Negation $a' = 1 - a$
  - Conjunction $a \wedge b = \min(a, b)$
  - Disjunction $a \vee b = \max(a, b)$
  - Implication $a \rightarrow b = \min(1, 1+b-a)$

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Apparatus of fuzzy logic is built on:

√ Fuzzy sets: describe the value of variables
√ Possibility distributions: constraints on the value of a variable
⇒ Linguistic variables: qualitatively and quantitatively described by fuzzy sets
Fuzzy if-then rules: knowledge

Linguistic variables

- Linguistic variable is "a variable whose values are words or sentences in a natural or artificial language". Each linguistic variable may be assigned one or more linguistic values, which are in turn connected to a numeric value through the mechanism of membership functions.
- Motivation: Conventional techniques for system analysis are intrinsically unsuited for dealing with systems based on human judgment, perception & emotion

Linguistic variables

- Principle of incompatibility: As the complexity of a system increases, our ability to make precise & yet significant statements about its behavior decreases until a fixed threshold. Beyond this threshold, precision & significance become almost mutually exclusive characteristics [Zadeh, 1973]
- LV represented by a quintuple (x, T(x), U, G, M)
  - x: name of variable
  - T(x): set of linguistic terms which can be a value of the variable
  - U: set of universe of discourse which defines the characteristics of the variable
  - G: syntactic grammar which produces terms in T(x)
  - M: semantic rules which map terms in T(x) to fuzzy sets in U

Linguistic variables

X = (Age, T(Age), U, G, M)

- Age: name of the variable X
- T(Age): {young, very young, very very young, …}
- U: [0,100] universe of discourse
- G(Age): $T^{i+1} = \{young\} \cup \{very\ T^i\}$
- M(young) = $\{(u, \mu_{young}(u)) \mid u \in [0,100]\}$

$$\mu_{young}(u) = \begin{cases} 1 & \text{if } u \in [0,25] \\ \left(1 + \dfrac{u-25}{5}\right)^{-2} & \text{if } u \in [25,100] \end{cases}$$

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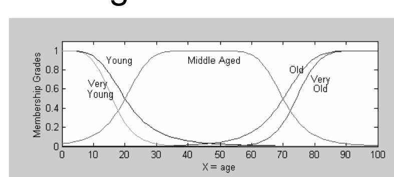
Linguistic variables



An example of a fuzzy linguistic variable and membership functions

Linguistic variables

- Fuzzy linguistic terms often consist of two parts:

1) Fuzzy predicate (primary term): expensive, old, rare, dangerous, good, etc.
2) Fuzzy modifier: very, likely, almost impossible, extremely unlikely, etc. Makes a composite linguistic term out of the primary term.

The modifier is used to change the meaning of predicate and it can be grouped into the following two classes:

a) Fuzzy truth qualifier or fuzzy truth value: quite true, very true, more or less true, mostly false, etc.

b) Fuzzy quantifier: many, few, almost, all, usually, etc.

Fuzzy predicate

- Fuzzy predicate
  – If the set defining the predicates of individual is a fuzzy set, the predicate is called a fuzzy predicate

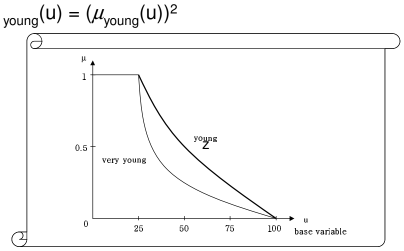
Example
- "z is expensive."
- "w is young."
- The terms "expensive" and "young" are fuzzy terms. Therefore the sets "expensive(z)" and "young(w)" are fuzzy sets

Fuzzy predicate

- When a fuzzy predicate "*x* is P" is given, we can interpret it in two ways
  – P(*x*) is a fuzzy set. The membership degree of *x* in the set P is defined by the membership function $\mu_{P(x)}$
  – $\mu_{P(x)}$ is the satisfactory degree of *x* for the property P. Therefore, the truth value of the fuzzy predicate is defined by the membership function
    - Truth value = $\mu_{P(x)}$

Fuzzy modifier (hedge)

- A new term can be obtained when we add a modifier "very" to a primary term
  – $\mu_{\text{very young}}(u) = (\mu_{\text{young}}(u))^2$

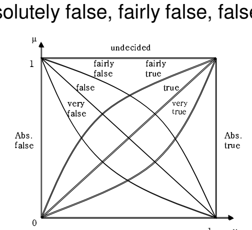


Fuzzy Truth Qualifier

- Baldwin defined fuzzy truth qualifier terms in the universal set V = {ν | ν ∈ [0,1]} as follows

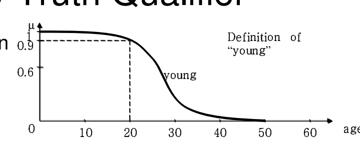
T = {true, very true, fairly true, absolutely true, … , absolutely false, fairly false, false}

$$\mu_{true}(v) = v$$
$$\mu_{very\ true}(v) = (\mu_{true}(v))^2$$
$$\mu_{fairly\ true}(v) = (\mu_{true}(v))^{1/2}$$
$$\mu_{false}(v) = 1 - \mu_{true}(v)$$
$$\mu_{very\ false}(v) = (\mu_{false}(v))^2$$
$$\mu_{fairly\ false}(v) = (\mu_{false}(v))^{1/2}$$

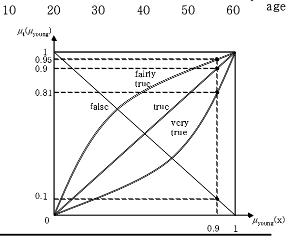


Fuzzy Truth Qualifier

- Consider a proposition P "20 is young". Truth value of this proposition is 0.9.



- Consider
  P$_1$="20 is young is true"
  P$_2$="20 is young is fairly true"
  P$_3$="20 is young is very true"
  P$_4$="20 is young is false"



Fuzzy Truth Qualifier

Example: Lets define membership functions "young" and "old" of linguistic variable "age"

$$\mu_{young}(x) = bell(x,20,2,0) = \frac{1}{1+\left(\frac{x}{20}\right)^4}$$

$$\mu_{old}(x) = bell(x,30,3,100) = \frac{1}{1+\left(\frac{x-100}{30}\right)^6}$$

Where x is the age of a person in the universe of discourse [0, 100]

Fuzzy Truth Qualifier

- More or less old = √old = $\int_X \sqrt{\frac{1}{1+\left(\frac{x-100}{30}\right)^6}} / x$

- Not young and not old = ¬young ∩ ¬old =

$$\int_X \left[1-\frac{1}{1+\left(\frac{x}{20}\right)^4}\right] \wedge \left[1-\frac{1}{1+\left(\frac{x-100}{30}\right)^6}\right] / x$$

Fuzzy Truth Qualifier

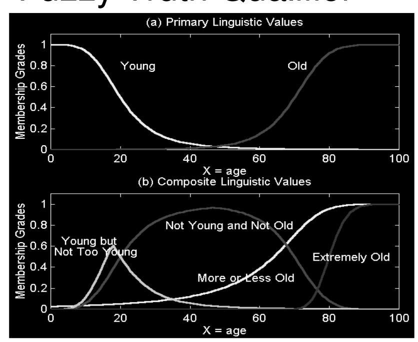
Young but not too young =
young ∩ ¬young² (too = very) =

$$\int_x \left[\frac{1}{1+\left(\frac{x}{20}\right)^4}\right] \wedge \left[1-\left(\frac{1}{1+\left(\frac{x}{20}\right)^4}\right)^2\right] / x$$

- Extremely old ≡ very very very old =

$$\int_x \left[\frac{1}{1+\left(\frac{x-100}{30}\right)^6}\right]^8 / x$$

Fuzzy Truth Qualifier



SUMMARY

- Fuzzy logic is based on predicate logic, where predicates are fuzzy
- Linguistic variables are variable taking linguistic values. LVs form the foundation of approximate reasoning in fuzzy logic.
- Fuzzy predicates can be formed from a primary term and a fuzzy modifier (hedge).