Optimal Control (Unit 3/CY/E0)

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Contents

Introduction to static optimisation. Dynamic optimisation. Discrete time systems. Continuous time systems. Open final time. Pontryagin's minimum principle. Tracking optimal control. LQG optimal control. Kalman filtering. Predictive control. Numerical methods.

Lecture notes, etc

http://www.rdg.ac.uk/~shs99vmb/notes

Recommended books

Lewis F.L. and Syrmos V.L. (1995) *Optimal Control*. Second Edition. Wiley.

Burl J.B. (2000). *Linear Optimal Control*. Addison-Wesley.

Bryson A.E. (2000). *Dynamic Optimization*. Addison-Wesley.

Assessment

Exam (100%)

Introduction

What is optimal control?

Optimal control is the process of finding control and state histories for a dynamic system over a period of time to minimise a performance index.

Optimal control is used in many fields, for example:

- To determine efficient maneuvers of aircraft, spacecraft and robots.

- To design feedback controllers that are optimal in some sense.

Introduction to static optimization

In order to learn the principles of optimal control, it is important to understand the theory of static optimisation.

Unconstrained optimization

A simple type of static optimization problem is to find the values of n variables $x_1, x_2, \cdots, x_n$ to minimise a function of these variables:

$$L(x_1, x_2, \cdots, x_n) \tag{1}$$

For convenience, vector notation is used, so let:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \tag{2}$$

then the optimisation problem is written as follows:

$$\min_x L(x) \tag{3}$$

Assuming that function L has first and second partial derivatives, then using Taylor series the function can be approximated about $\bar{x}$:

$$L(x) \approx L(\bar{x}) + L_x(x-\bar{x}) + \frac{1}{2}(x-\bar{x})L_{xx}(x-\bar{x}) \quad (4)$$

where the partial derivatives $L_x = \partial L/\partial x$ (row vector) and $L_{xx} = \partial^2 L/\partial x^2$ (a $n \times n$ matrix called the Hessian) are computed at $\bar{x}$.

The values at which $L_x = 0$ are called stationary points.

From the approximation, we can infer that the necessary conditions for a minimum are:

$$\begin{aligned} L_x &= 0 \\ L_{xx} &\geq 0 \end{aligned} \tag{5}$$

whereas sufficient conditions for a minimum are:

$$\begin{aligned} L_x &= 0 \\ L_{xx} &> 0 \end{aligned} \tag{6}$$

When a matrix satisfies the relation > 0 (≥ 0) we say that the matrix is positive definite (positive semi–definite). A positive definite matrix has positive eigenvalues.

Example 1.1

$$L(x) = \frac{1}{2}[x_1\, x_2] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [b_1\, b_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7}$$

Or, in matrix-vector notation:

$$L(x) = \frac{1}{2} x^T Q x + b^T x \tag{8}$$

A stationary point is given by:

$$L_x = Qx^* + b = 0 \Rightarrow x^* = -Q^{-1} b \tag{9}$$

The type of stationary point depends on the Hessian $L_{xx} = Q$

- If $L_{xx} > 0$ we have a minimum.

- If $L_{xx} < 0$ we have a maximum.

- If $L_{xx} \geq$ or $L_{xx} \leq 0$ we have a singular point. We cannot say if the stationary point is a minimum or a maximum.

- If L_{xx} has both positive and negative eigenvalues, then we have a *saddle point*.

Optimisation with equality constraints

A more general class of static optimisation problem involves a set of p constraint relations:

$$f^{(i)}(x) = 0,\ i = 1, \cdots, p \qquad (10)$$

where $p < n$.

Using vector notation, let

$$f = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(p)} \end{bmatrix} \qquad (11)$$

Then the static optimisation problem can be written as follows:

$$\min_x L(x) \tag{12}$$

subject to

$$f(x) = 0 \tag{13}$$

Suppose we have a feasible x, such that $f(x) = 0$. We wish to analyse the conditions for which x provides a minimum for $L(x)$.

We look for an infinitesimal change in x that will decrease L while keeping $f = 0$ to first order in dx:

$$\begin{aligned} dL &= L_x \; dx < 0 \\ df &= f_x \; dx = 0 \end{aligned} \tag{14}$$

where $f_x = \partial f / \partial x$ is a $p \times n$ matrix known as the Jacobian. The rows of f_x are the gradients of $f^{(i)}$ with respect to x, $i = 1, \cdots, n$. These gradients are perpendicular to the curve $f = 0$.

At a minimum, L_x must have no component parallel to the curve $f = 0$; otherwise a dx could be found to make $dL < 0$ while keeping $df = 0$. Then L_x must be perpendicular to $f = 0$ at a minimum.

Then, L_x must be a linear combination of the n constraint gradients $f_x^{(i)}$:

$$L_x = -\sum_{i=1}^{n} \lambda_i f_x^{(i)} \tag{15}$$

where λ_i is a set of constants known as the *Lagrange multipliers*. In vector notation:

$$L_x = -\lambda^T f_x \tag{16}$$

where

$$\lambda^T = [\lambda_1, \cdots, \lambda_n] \tag{17}$$

Then, the necessary conditions for a stationary point are:

$$f(x) = 0 \tag{18}$$

$$L_x + \lambda^T f_x = 0 \tag{19}$$

which are $n + p$ equations for the $n + p$ unknowns x and λ.

Define a function H as follows:

$$H \stackrel{\triangle}{=} L + \lambda^T f \tag{20}$$

Then, optimality condition (19) can be written as follows:

$$H_x = 0 \tag{21}$$

Example 1.2

$$\min_x L = x_1^2 + x_2^2$$

subject to:

$$2x_1 + x_2 + 4 = 0$$

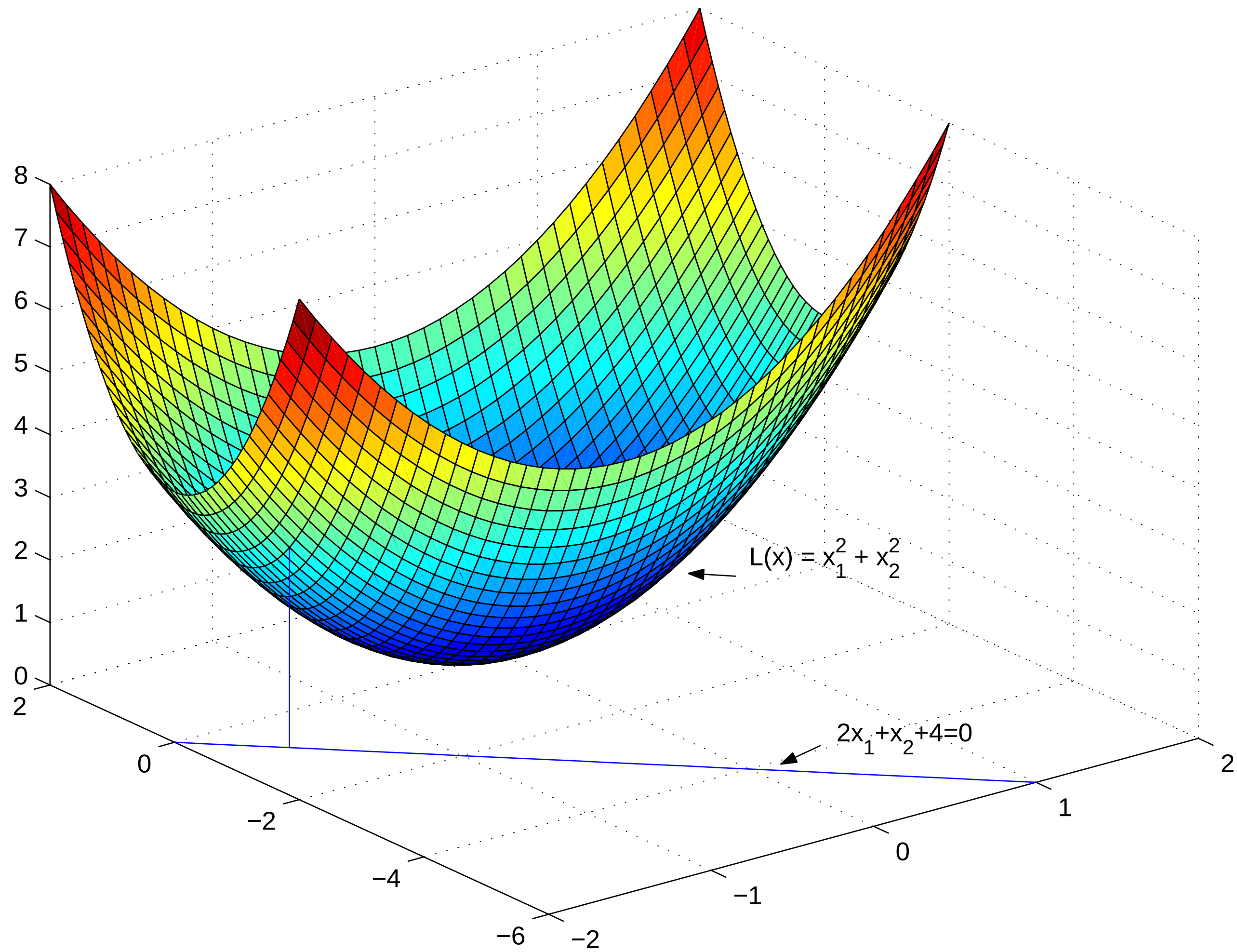
Solution

$$H = x_1^2 + x_2^2 + \lambda(2x_1 + x_2 + 4)$$

Optimality conditions:

$$\begin{array}{ll} f(x) = 0 & \Rightarrow 2x_1 + x_2 + 4 = 0 \\ H_x = 0 & \Rightarrow \left\{ \begin{array}{ll} 2x_1 + 2\lambda & = 0 \\ 2x_2 + \lambda & = 0 \end{array} \right. \end{array}$$

Solving for the unknowns results in $x_1 = -1.6$, $x_2 = -0.8$, and $\lambda = 1.6$



Example 1.2 illustrated