Dynamic Optimisation

Discrete Dynamic Systems

A single stage example

Suppose that we have a specific single stage dynamic system governed by the following equation:

$$x_1 = ax_0 + bu_0,\ x_0 = x_i \tag{1}$$

where x is a scalar state and u is a scalar control input.

We wish to minimise the following objective, subject to the dynamics of the system:

$$\min_{u_0} J = x_1^2 \tag{2}$$

Substituting x_1 to include the system constraint:

$$\min_{u_0} J = (ax_0 + bu_0)^2 = a^2x_0^2 + b^2u_0^2 + 2abx_0u_0 \tag{3}$$

We can find $\partial J/\partial u_0$ and make it zero:

$$\frac{\partial J}{\partial u_0} = 2b^2u_0 + 2abx_0 = 0 \tag{4}$$

Therefore:

$$u(0) = -(a/b)x_0 \tag{5}$$

which gives:

$$x_1 = ax_0 + b(-a/b)x_0 = 0 \tag{6}$$

Therefore the optimal control $u_0 = -(a/b)x_0$ gives the minimun value of the objective function $J = 0$.

A two stage example

Now, take the same system over two stages:

$$\begin{aligned} x_0 &= x_i \\ x_1 &= ax_0 + bu_0 \\ x_2 &= ax_1 + bu_1 = a^2x_0 + abu_0 + bu_1 \end{aligned} \tag{7}$$

And assume that we wish to minimise the following objective function:

$$\min_{u_0,u_1} J = x_2^2 + u_1^2 + u_0^2 \tag{8}$$

Substituting x_2, we have

$$\min_{u_0,u_1} J = (a^2x_0 + abu_0 + bu_1)^2 + u_1^2 + u_0^2 \quad (9)$$

The partial derivatives of J with respect to u_0 and u_1 are:

$$\frac{\partial J}{\partial u_0} = 2ab(a^2x_0 + abu_0 + bu_1) + 2u_0 = 0 \quad (10)$$

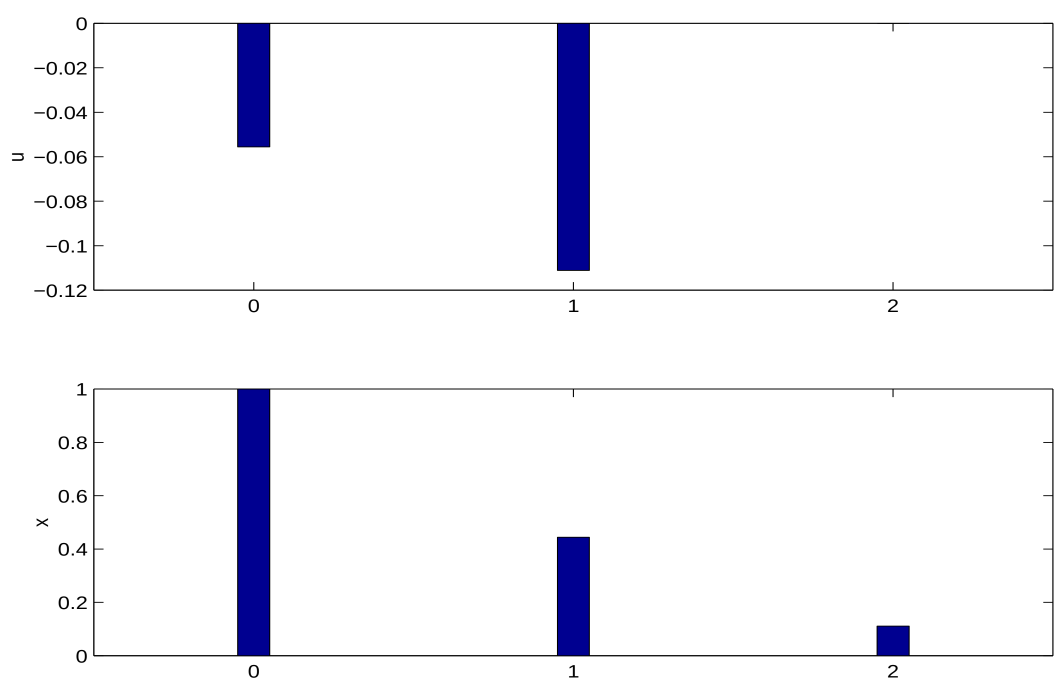
$$\frac{\partial J}{\partial u_1} = 2b(a^2x_0 + abu_0 + bu_1) + 2u_1 = 0 \quad (11)$$

We have two linear equations with two unknows: u_0 and u_1. The solution is:

$$\begin{aligned} u_0 &= \frac{-ba^3}{ba^2+b^2+1}x_0 \\ u_1 &= \frac{-ba^2}{ba^2+b^2+1}x_0 \end{aligned} \quad (12)$$

Suppose that we know that $a = 0.5$, $b = 1$ and $x_0 = 1$. Then we have:

$$
\begin{aligned}
u_0 &= -0.11 \\
u_1 &= -0.0556 \\
x_1 &= 0.4444 \\
x_2 &= 0.1111 \\
J &= 0.0278
\end{aligned}
\tag{13}
$$



General discrete case

Suppose that a dynamic system is described by the following equation which determines the transition from the n-dimensional state x_k to state x_{k+1}, given the m-dimensional control vector u_k:

$$x_{k+1} = f(x_k, u_k, k),\ x_0 = x_i \qquad (14)$$

A fairly general optimisation problem for such systems is to find the sequence of controls $u_k, k = 0, ...N-1$ to minimise a performance index of the form:

$$J = \phi\left(x_N\right) + \sum_{k=0}^{N-1} L\left(x_k, u_k, k\right) \qquad (15)$$

subject to

$$x_{k+1} = f(x_k, u_k, k),\ x_0 = x_i \qquad (16)$$

This is an optimisation problem with equality constraints.

Necessary optimality conditions

Adjoin the constraints to the performance index with a sequence of Lagrange multiplier vectors λ_k as follows:

$$\bar{J} = \phi(x_N) + \lambda_0^T [x_i - x_0] + \sum_{k=0}^{N-1} \left\{ L_k + \lambda_{k+1}^T \left[f_k - x_{k+1} \right] \right\} \quad (17)$$

Define the Hamiltonian as follows:

$$H_k = L(x_k, u_k, k) + \lambda_{k+1}^T f(x_k, u_k, k) \quad (18)$$

So that

$$\bar{J} = \phi\left(x_N\right) - \lambda_N^T x_N + \lambda_0^T x_i + \sum_{k=0}^{N-1} \left\{ H_k - \lambda_k^T x_k \right\} \tag{19}$$

The optimality conditions are then found using optimisation theory, which involves calculating the increment $d\bar{J}$ and making it zero.

The optimality conditions are:

$$x_{k+1} = f(x_k, u_k, k) \tag{20}$$

$$\lambda_k = \frac{\partial H}{\partial x_k}^T \tag{21}$$

$$\frac{\partial H}{\partial u_k}^T = 0 \tag{22}$$

The boundary conditions are:

$$x_0 = x_i \tag{23}$$

$$\lambda_N = \frac{\partial \phi}{\partial x_N}^T \tag{24}$$

We have a two point boundary value problem.

Discrete Linear-Quadratic Regulator

Let the plant to be controlled be described by the linear equation

$$x_{k+1} = Ax_k + Bu_k \tag{25}$$

Suppose that we wish to minimise the following quadratic performance index:

$$J = \tfrac{1}{2}x_N^T S x_N + \tfrac{1}{2}\sum_{k=0}^{N-1}\left[x_k^T Q x_k + u_k^T R u_k\right] \tag{26}$$

We assume that Q and S are positive semidefinite matrices and that R is positive definite.

In this case, the Hamiltonian is given by:

$$H_k = \frac{1}{2}\left(x_k^T Q x_k + u_k^T R u_k\right) + \lambda_{k+1}^T \left(A x_k + B u_k\right) \tag{27}$$

From the necessary optimality conditions, we have:

$$x_{k+1} = A x_k + B u_k \tag{28}$$

$$\lambda_k = \frac{\partial H}{\partial x_k}^T = Q x_k + A^T \lambda_{k+1} \tag{29}$$

$$\frac{\partial H}{\partial u_k}^T = R u_k + B^T \lambda_{k+1} = 0 \tag{30}$$

From (30) we can obtain the optimal control:

$$u_k = -R^{-1}B^T\lambda_{k+1} \tag{31}$$

The boundary conditions are:

$$\begin{array}{ll} x_0 & \text{specified} \\ \lambda_N & = Sx_N \end{array} \tag{32}$$

We have a linear two point boundary value problem.

Riccati Solution

The solution to the linear two point boundary value problem can be found by solving backwards from $S_N = S$, the following Riccati equation:

$$\begin{aligned} S_k &= A^T[S_{k+1} - \\ &\quad S_{k+1}B(B^T S_{k+1} B + R)^{-1} B^T S_{k+1}]A + Q \end{aligned} \tag{33}$$

The state feedback gain matrix is given by:

$$K_k = \left(B^T S_{k+1} B + R_k\right)^{-1} B^T S_{k+1} A \tag{34}$$

The optimal control is:

$$u_k = -K_k x_k \tag{35}$$

and the optimal state is:

$$x_{k+1} = (A - BK_k)\, x_k \tag{36}$$

Steady State Solutions

When the number of samples N approaches infinity, then under certain conditions the Riccati solution converges to a fixed values of S and K.

$$S = A^T \left[S - SB(B^T SB + R)^{-1} B^T S \right] A + Q \tag{37}$$

$$K = \left(B^T SB + R \right)^{-1} B^T SA \tag{38}$$

Equation (37) is known as the discrete Riccati Algebraic Equation (RAE).

In this case, the optimal control is:

$$u_k = -K x_k \tag{39}$$

and the optimal state is:

$$x_{k+1} = (A - BK)\, x_k \tag{40}$$

Example: LQ regulation of an unstable scalar system.

Consider the unstable system:

$$x_{k+1} = 2x_k + u_k \tag{41}$$

Assume that we wish to regulate this system using steady state LQ control, given the following performance index:

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left[x_k^2 + 2u_k^2 \right] \tag{42}$$

The algebraic Riccati equation becomes:

$$S^2 - 7S - 2 = 0 \tag{43}$$

which has the solutions $S_1 = 7.2749$ and $S_2 = -0.2749$.

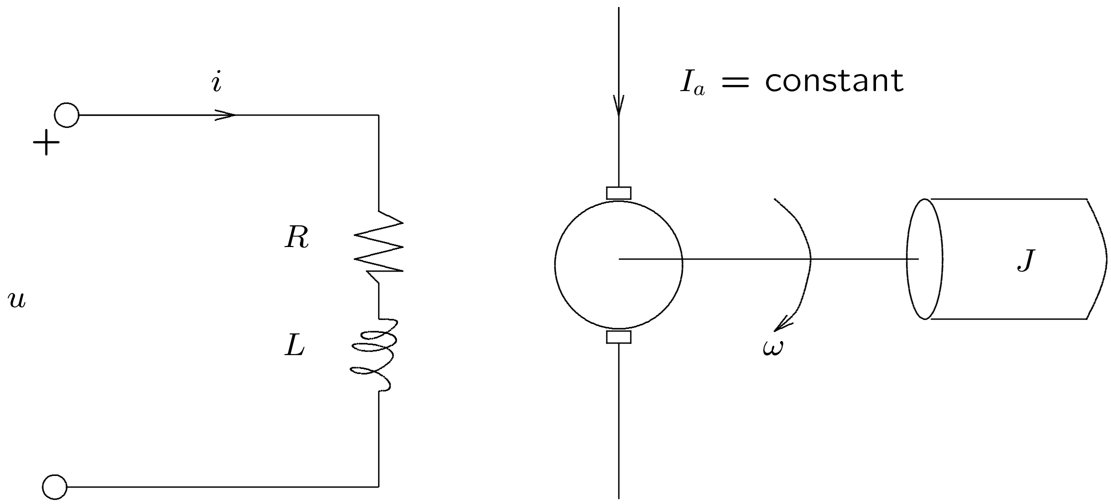
Taking the positive solution, gives the following state feedback law:

$$u_k = -1.5687x_k \tag{44}$$

and the closed loop system becomes stable:

$$x_{k+1} = 2x_k + (-1.5687x_k) = -0.4313x_k \tag{45}$$

Example: DC motor under state feedback



The state equations for a dc motor with constant armature current are:

$$\frac{di(t)}{dt} = -\frac{R}{L}i(t) + \frac{1}{L}u(t) \tag{46}$$

$$\frac{d\omega(t)}{dt} = \frac{K}{J}i(t) \tag{47}$$

where $K = k_t i_a$, and J is the moment of inertia. Assuming $R = L = K = 1$, find a discrete time state feedback controller with sampling time $h = 1$ to minimise the following performance index:

$$J = \sum_{k=0}^{\infty} i_k^2 + \omega_k^2 + u_k^2 \tag{48}$$

Solution:

Continuous time model:

$$\frac{d}{dt}\begin{bmatrix} i \\ \omega \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} i \\ \omega \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\bar{B}} u \qquad (49)$$

Discrete time model using zero–order–hold discretisation:

$$\begin{bmatrix} i_{k+1} \\ \omega_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.368 & 0 \\ 0.632 & 0 \end{bmatrix}}_{e^{\bar{A}h}} \begin{bmatrix} i_k \\ \omega_k \end{bmatrix} + \underbrace{\begin{bmatrix} 0.632 \\ 0.368 \end{bmatrix}}_{\int_0^h e^{\bar{A}h} dt \bar{B}} u_k \qquad (50)$$

Riccati solution using Matlab:

K = dlqr(A, B, Q, R)

$$K = [\,0.615 \quad 0.628] \qquad (51)$$

Control law:

$$u_k = 0.615 i_k + 0.628 \omega_k \qquad (52)$$

Suppose that the system starts from the initial condition $i(0) = 1$ A and $\omega(0) = 2$ rad/s.