Continuous Dynamic Systems

Optimal control problems for continuous dynamic systems are problems in the calculus of variations.

They may be considered as limiting cases of discrete optimal control problems for very small time steps.

A continuous time dynamic system is described by a set of differential equations:

$$\dot{x}(t) = f(x(t), u(t), t) \qquad (1)$$

Where $x(t)$ is a n-dimensional state vector and $u(t)$ is a m-dimensional control vector.

Introduction to calculus of variations

Definition: A functional J is a rule of correspondence that assigns to each function x in a certain class Ω a unique real number.

Example: Suppose that x is a continuous function of t defined in the interval $[t_0, t_f]$ and

$$J(x) = \int_{t_0}^{t_f} x(t)\, dt \tag{2}$$

is the real number assigned by the functional J (in this case, J is the area under the $x(t)$ curve).

Definition: The increment of a funcional J is defined as follows:

$$\Delta J = J(x + \delta x) - J(x) \qquad (3)$$

where δx is called the *variation* of x

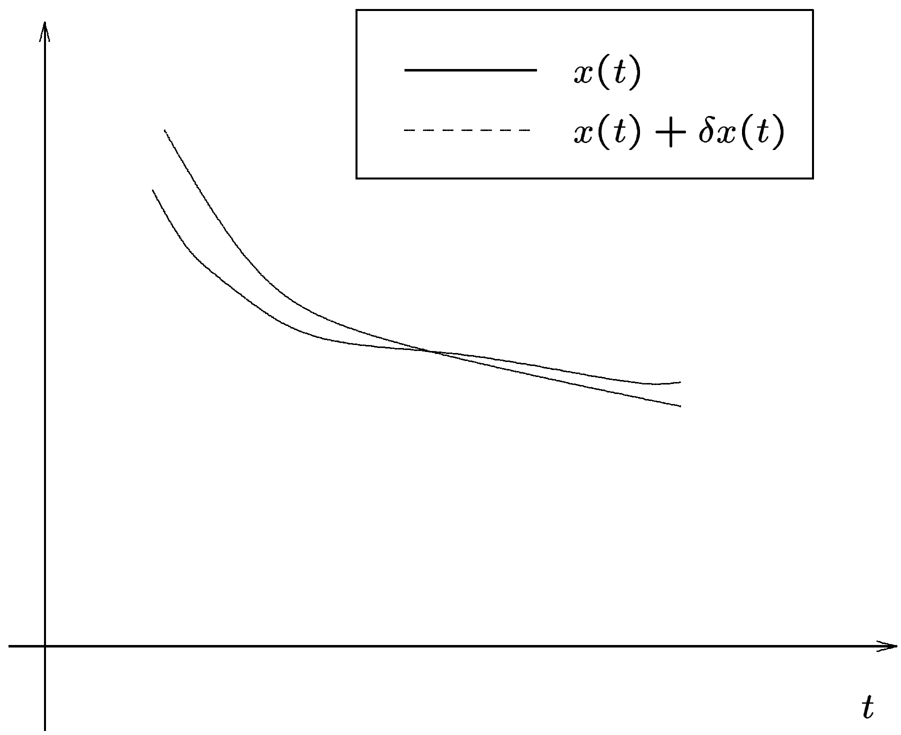


Figure 3.1: Illustration of the variation of x.

Definition: The variation of a differentiable functional J is defined as follows:

$$\delta J = \lim_{||\delta x|| \to 0} \Delta J = \lim_{||\delta x|| \to 0} J(x + \delta x) - J(x) \tag{4}$$

Example: Find the variation δJ of the following functional:

$$J(x) = \int_0^1 x^2(t)\, dt \tag{5}$$

Solution:

$$\Delta J = J(x+\delta x) - J(x) = \int_0^1 (x+\delta x)^2 dt - \int_0^1 x^2 dt \tag{6}$$

$$\Delta J = \int_0^1 (x^2 + 2x\delta x - x^2) dt + \int_0^1 [\delta x]^2 dt \tag{7}$$

$$\Delta J = \int_0^1 (2x\delta x) dt + \int_0^1 [\delta x]^2 dt \tag{8}$$

$$\delta J = \lim_{||\delta x|| \to 0} \left[\int_0^1 (2x\delta x) dt + \underbrace{\int_0^1 [\delta x]^2 dt}_{\text{neglected}} \right] \tag{9}$$

$$\delta J = \int_0^1 (2x\delta x)\, dt \tag{10}$$

In general, the variation of a differentiable functional of the form:

$$J = \int_{t_0}^{t_f} L(x)dt \tag{11}$$

can be calculated as follows:

$$\delta J = \int_{t_0}^{t_f} \left\{ \left[\frac{\partial L(x)}{\partial x} \right] \delta x \right\} dt \tag{12}$$

Definition: A functional J has a local minimum at x^* if for all functions x in the vecinity of x^* we have:

$$\Delta J = J(x) - J(x^*) \geq 0 \tag{13}$$

For a local maximum, the condition is:

$$\Delta J = J(x) - J(x^*) \leq 0 \tag{14}$$

Important result. If x^* is a local minimum or maximum, then the variation of a functional J must vanish on x^*:

$$\delta J(x^*, \delta x) = 0 \tag{15}$$

Continuous optimal control

A continuous time optimal control problem can be written as follows:

$$\min_{u(t)} J = \phi(x(t_f)) + \int_{t_o}^{t_f} L(x,u,t)dt \qquad (16)$$

Subject to:

$$\dot{x} = f(x,u,t) \qquad (17)$$

with t_o, t_f and $x(t_o)$ specified.

Where J is a scalar performance index, ϕ is a terminal weighting function and L is a weighting function.

Necessary Optimality Conditions

Adjoin the constraints to the performance index with a time–varying Lagrange multiplier vector:

$$\begin{aligned}\bar{J} = \phi(x(t_f)) + \textstyle\int_{t_o}^{t_f} \{L(x,u,t) \\ +\lambda^T(t)\left[f(x,u,t) - \dot{x}\right]\Big\}\, dt\end{aligned} \tag{18}$$

Define the scalar Hamiltonian function

$$H(t) = L(x,u,t) + \lambda^T(t) f(x,u,t) \tag{19}$$

So that

$$\bar{J} = \phi(x(t_f) + \int_{t_o}^{t_f} \left\{H(t) - \lambda^T(t)\dot{x}\right\} dt \tag{20}$$

Now consider an infinitesimal variation in $u(t)$, that we will call $\delta u(t)$. Such a variation will produce variations in the state history $\delta x(t)$, and a variation in the performance index $\delta \bar{J}$:

$$\delta \bar{J} = \left[(\frac{\partial \phi}{\partial x} - \lambda^T)\delta x \right]_{t=t_f} + \left[\lambda^T \delta x \right]_{t=t_O} + \int_{t_o}^{t_f} \left\{ \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \delta x + \left(\frac{\partial H}{\partial u} \right) \delta u \right\} dt \tag{21}$$

So, to make $\delta \bar{J} = 0$ we have

$$\left[\frac{\partial H}{\partial u} \right]^T = 0, \; t_o \leq t \leq t_f \tag{22}$$

$$\lambda^T(t_o)\delta x(t_o) = 0 \tag{23}$$

$$\dot{\lambda} = - \left[\frac{\partial H}{\partial x} \right]^T \tag{24}$$

If $x(t_o)$ is fixed, then $\delta x(t_o) = 0$

The boundary conditions are

$$x(t_o) \text{ specified}$$

$$\lambda(t_f) = \left[\frac{\partial \phi}{\partial x}\right]^T \Bigg|_{t=t_f} \tag{25}$$

Example: Temperature control in a room

It is desired to heat a room using the least possible energy. The temperature dynamics of the room are:

$$\dot{\theta} = -2(\theta - \theta_a) + u, \ \theta(0) = \theta_a \qquad (26)$$

where θ is the room temperature, θ_a is the external air temperature, $u(t)$ is the rate of heat supply to the room and t is the time in hours.

Redefine the state as $x = \theta - \theta_a$, the the room dynamics can be written as follows:

$$\dot{x} = -2x + u, \ x(0) = 0 \qquad (27)$$

In order to bring the state close to a desired value $x_d = 10$ at the end of the fixed interval $t \in [0, 1]$ with minimum energy, we are asked to minimise the following performance index:

$$J = \frac{1}{2} 10(x(1) - 10)^2 + \frac{1}{2} \int_0^1 u^2(t) dt \qquad (28)$$

Solution:

The Hamiltonian is:

$$H = \frac{u^2}{2} + \lambda(-2x + u) \tag{29}$$

Applying the optimality conditions, we have:

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u + \lambda = 0 \Rightarrow u = -\lambda \tag{30}$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} \Rightarrow \dot{\lambda} = -(-2\lambda) \Rightarrow \dot{\lambda} = 2\lambda \tag{31}$$

The boundary condition for λ is:

$$\lambda(1) = \left.\frac{\partial \phi}{\partial x}\right|_{t=1} = 10(x(1) - 10) \tag{32}$$

From Eq. (31), we have:

$$\lambda(t) = \lambda(1)e^{-2(1-t)} \tag{33}$$

Therefore the optimal control has the form:

$$u(t) = -\lambda(1)e^{-2(1-t)} \tag{34}$$

Replacing this into the state equation:

$$\dot{x} = -2x - \lambda(1)e^{-2(1-t)} = -2x - Fe^{2t} \tag{35}$$

where $F = \lambda(1)e^{-2}$. Taking the Laplace transform to solve this differential equation:

$$sX(s) - \underbrace{x(0)}_{=0} = -2X(s) - F\frac{1}{s-2} \tag{36}$$

so that:

$$X(s) = -F\frac{1}{(s^2 - 2^2)} \tag{37}$$

The solution is then:

$$x(t) = -\frac{F}{2}\sinh(2t) = -\frac{\lambda(1)e^{-2}}{2}\sinh(2t) \tag{38}$$

Evaluating the above equation at $t = 1$, we obtain $x(1) = -0.245\lambda(1)$, and replacing in Eq. (32),

$$\lambda(1) = 10(-0.245\lambda(1) - 10) \tag{39}$$

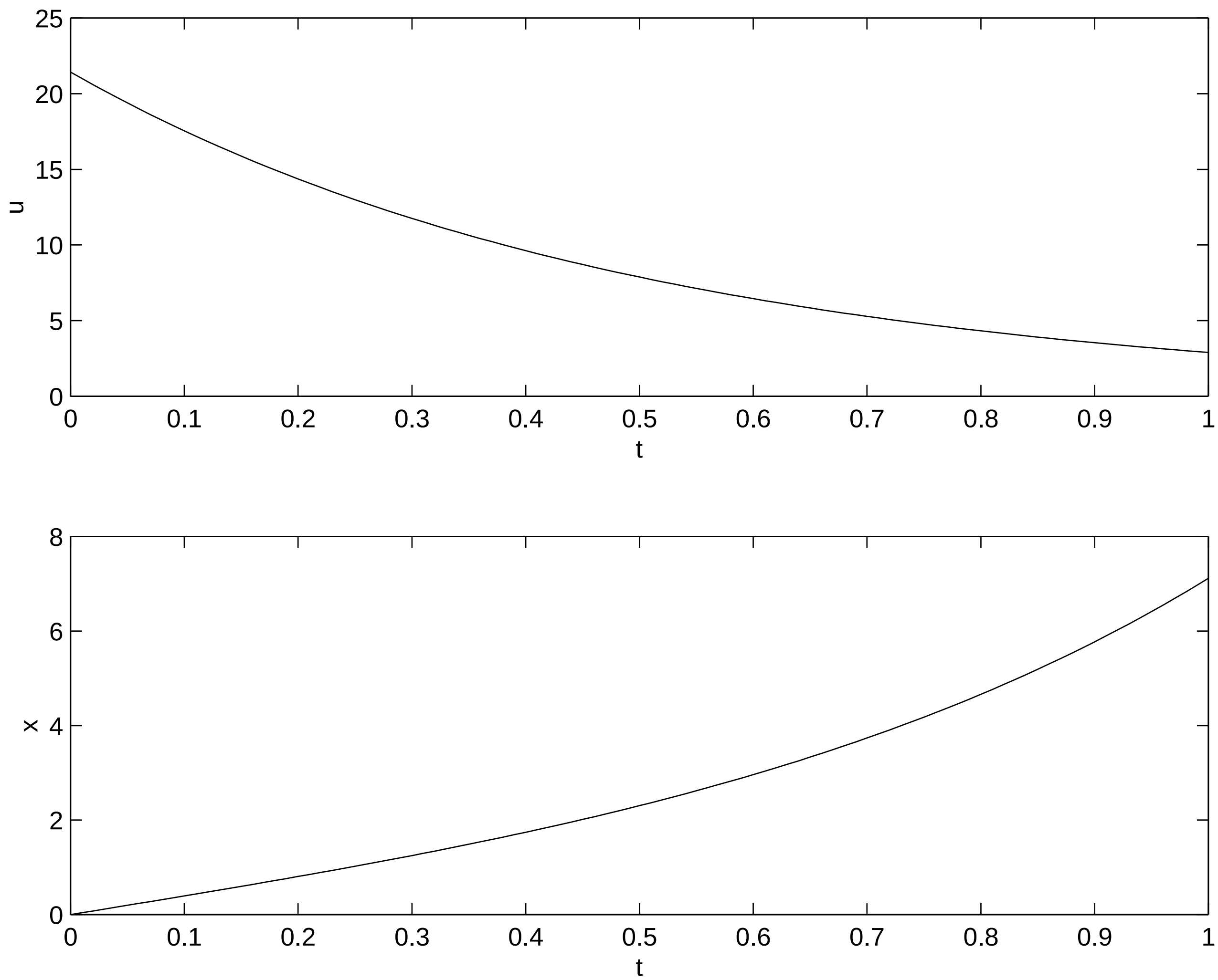
which gives $\lambda(1) = -28.99$

So that the optimal control is:

$$u(t) = 28.99e^{-2(1-t)} \tag{40}$$

and the optimal state is:

$$x(t) = 1.962 \sinh(2t) \tag{41}$$



Continuous LQ Regulator

$$\min_{u(t)} J = x(t_f)^T S(t_f) x(t_f) + \frac{1}{2}\int_{t_o}^{t_f} (x^T Q x + u^T R u) dt \tag{42}$$

subject to:

$$\dot{x}(t) = Ax(t) + Bu(t),\ t_0 \leq t \leq t_f \tag{43}$$

Assumptions: $S(t_f) \geq 0,\ Q \geq 0,\ R > 0$

Hamiltonian:

$$H = \frac{1}{2}(x^T Q x + u^T R u) + \lambda^T (Ax + Bu) \tag{44}$$

Optimality conditions

$$\left[\frac{\partial H}{\partial u}\right]^T = 0 \Rightarrow u(t) = -R^{-1}B^T\lambda(t) \qquad (45)$$

$$\dot{\lambda}(t) = -\left[\frac{\partial H}{\partial x}\right]^T \Rightarrow \dot{\lambda} = -Qx(t) - A^T\lambda(t) \qquad (46)$$

Boundary conditions:

$$\lambda(t_f) = \left[\frac{\partial \phi(x(t_f))}{\partial x}\right]^T \Rightarrow \lambda(t_f) = S(t_f)x(t_f) \qquad (47)$$

$$x(t_0) = x_0 \qquad (48)$$

Riccati solution:

$$-\dot{S} = A^T S + SA - SBR^{-1}B^T S + Q,\ S(t_f) \text{ given} \tag{49}$$

$$K(t) = R^{-1}B^T S(t) \tag{50}$$

$$u(t) = -K(t)x(t) \tag{51}$$

LQ case with infinite horizon

$$\min_{u(t)} J = \frac{1}{2}\int_0^{\infty}(x^TQx + u^TRu)dt \tag{52}$$

subject to:

$$\dot{x}(t) = Ax(t) + Bu(t),\ 0 \leq t < \infty \tag{53}$$

Assumptions: $Q \geq 0$, $R > 0$

Riccati solution: Continuous Algebraic Riccati Equation

$$0 = A^TS + SA - SBR^{-1}B^TS + Q \tag{54}$$

$$K = R^{-1}B^TS, \text{ a constant matrix} \tag{55}$$

$$u(t) = -Kx(t) \tag{56}$$

Using Matlab we can easily find the optimal state feedback gain using the following command:

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K = lqr( A, B, Q, R )
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Example: Consider the unstable system:

$$\dot{x} = 2x + u \tag{57}$$

Find state the state feedback controller that minimises the following performance index:

$$J = \frac{1}{2}\int_0^\infty (x^2 + u^2)dt \tag{58}$$

Solution:

Algebraic Riccati equation:

$$-S^2 + 4S + 1 = 0 \tag{59}$$

The positive solution is: $S = 2.22$.

The state feedback gain is:

$$K = R^{-1}B^T S = 2.22 \tag{60}$$

So that the control law is:

$$u(t) = -2.22x(t) \tag{61}$$

and the closed loop system is stable:

$$\dot{x} = -0.22x(t) \tag{62}$$