Properties of LQ regulators

Stability of LQ regulators

One of the important properties of LQ regulators is that provided certain conditions are met, they guarantee nominally stable closed loop systems.

Recall the algebraic Riccati equation for continuous time systems:

$$0 = A^T S + SA - SBR^{-1}B^T S + Q \qquad (1)$$

The state feedback gain matrix is:

$$K = R^{-1}B^T S, \text{ a constant matrix} \qquad (2)$$

The control law is:

$$u(t) = -Kx(t) \qquad (3)$$

and the closed loop system is:

$$\dot{x} = (A - BK)x(t) \qquad (4)$$

Definition 1 (Stabilizability) *A state space pair* (A, B) *is stabilizable is there exists a state feedback gain* K *such that the closed loop system matrix* $(A - BK)$ *is stable.*

If a system is stabilizable then all unstable eigenvalues of the A matrix can be made stable by means of constant state feedback $(A - KB)$.

Notice that controllability implies stabilizability.

Definition 2 (Controllability) *The dynamic system* $\dot{x} = Ax + Bu$, *is said to be controllable if the controllability matrix:*

$$\mathcal{C} = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array} \right]$$

has rank n, *where* n *is the dimension of the state vector* x.

Definition 3 (Observability) *The dynamic system $\dot{x} = Ax + Bu$, $y = Cx$ is said to be observable if the observability matrix:*

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n, where n is the dimension of the state vector x.

The conditions for achieving a stable LQ system are as follows.

- $R > 0$, $Q \geq 0$
- (A, B) stabilizable
- (A, C) observable, where $Q = C^T C$

Example

Consider the problem of finding an optimal control law for a linear time invariant system with:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and a performance index:

$$J = \frac{1}{2}\int_0^{\infty} \left\{ 2x_1^2(t) + 2u^2(t) \right\} dt$$

Check if this system can be made stable by LQ control and if so find the optimal control law.

Solution

The system is open loop unstable, as it has two zero eigenvalues.

Checking controllability:

$$\mathcal{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \text{rank}(\mathcal{C}) = 2$$

The system is controllable and therefore it is stabilizable.

We have that

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 2$$

We need to find C such that $C^T C = Q$.

$$C = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{O} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, \ \text{rank}(\mathcal{O}) = 2$$

The system is observable through C. Then, we know that the closed loop optimal control system will be stable.

We will now solve the algebraic Riccati equation.

Define the symmetric matrix S as follows:

$$S = \begin{bmatrix} s_{11} & s_{21} \\ s_{21} & s_{22} \end{bmatrix}$$

and let's calculate all the terms of the ARE:

$$A^T S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_{11} & s_{21} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ s_{11} & s_{21} \end{bmatrix}$$

$$SA = \begin{bmatrix} s_{11} & s_{21} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & s_{11} \\ 0 & s_{21} \end{bmatrix}$$

$$\begin{aligned} -SBR^{-1}B^T S &= -\tfrac{1}{2} \begin{bmatrix} s_{11} & s_{21} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s_{11} & s_{21} \\ s_{21} & s_{22} \end{bmatrix} \\ &= -\tfrac{1}{2} \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} \begin{bmatrix} s_{21} & s_{22} \end{bmatrix} \\ &= -\tfrac{1}{2} \begin{bmatrix} s_{21}^2 & s_{21}s_{22} \\ s_{21}s_{22} & s_{22}^2 \end{bmatrix} \\ Q &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

This gives three equations:

$$-\frac{1}{2}s_{21}^2 + 2 = 0 \Rightarrow s_{21} = 2$$

$$-\frac{1}{2}s_{22}^2 + 2s_{21} = 0 \Rightarrow s_{22} = \sqrt{4s_{21}} = 2\sqrt{2}$$

$$-\frac{1}{2}s_{21}s_{22} + s_{11} = 0 \Rightarrow s_{11} = \frac{1}{2}s_{21}s_{22} = 2\sqrt{2}$$

The state feedback gain is:

$$R^{-1}B^TS = \frac{1}{2}\left[\begin{array}{cc} 0 & 1 \end{array}\right]\left[\begin{array}{cc} 2\sqrt{2} & 2 \\ 2 & 2\sqrt{2} \end{array}\right] = \left[\begin{array}{cc} 1 & \sqrt{2} \end{array}\right]$$

and the control law is:

$$u(t) = -x_1(t) - \sqrt{2}x_2(t)$$

The eigenvalues of the closed loop matrix $(A - KB)$ are: -0.65 and -2.17.

Tuning LQ regulators

Tuning LQ regulators implies choosing the weight matrices Q and R. This usually involves some kind of trial and error.

However, because LQ regulators provide stable closed loop systems, we do not have to worry about stability.

Q and R are usually chosen as diagonal matrices, so that for a system with n states and m controls we have $n+m$ parameters to choose.

The values of Q_{jj} and R_{jj} are chosen according to the *relative importance* of each state and control variable, bearing in mind that we must have $R_{jj} > 0$ and $Q_{jj} \geq 0$.

Tracking Optimal Control

The LQR controller is designed to drive the states to zero. This is a limitation since most control systems are tracking systems, where the output is forced to achieve a desired value.

Consider the system:

$$\dot{x} = Ax + Bu \tag{5}$$

and the output equation:

$$y = Cx \tag{6}$$

where y is the output of the system and C is a matrix. Assume that the output vector y has the same dimension as the input vector u.

In the steady state, we assume that for an output reference r there exists a combination of state $\bar{x}$ and input $\bar{u}$ that makes $y = r$. This implies that A must be invertible (no zero eigenvalues) and that the inverse of $CA^{-1}B$ must exist.

$$\dot{\bar{x}} = A\bar{x} + B\bar{u} = 0 \Rightarrow \bar{x} = -A^{-1}B\bar{u} \tag{7}$$

Substituting in 6:

$$\bar{y} = -CA^{-1}B\bar{u} = r \Rightarrow \bar{u} = -\left[CA^{-1}B\right]^{-1} r \tag{8}$$

Tracking via coordinate translation

An intuitive approach to make the output y follow a reference r it to replace the state equation in LQR with a state error equation.

Define $\tilde{x} = x - \bar{x}$ and $\tilde{u} = u - \bar{u}$. Then the errors satisfy:

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} \tag{9}$$

A suitable performance index would be:

$$J = \int_0^\infty \tilde{x}^T Q \tilde{x} + \tilde{u}^T R \tilde{u},\ \ Q \geq 0, R > 0 \quad (10)$$

If the pair (A, B) is stabilizable and $(A, Q^{1/2})$ is observable, the optimal feedback law is:

$$\tilde{u}(t) = -K\tilde{x}(t) \Rightarrow u(t) = \bar{u}(t) - K(x(t) - \bar{x}(t)) \quad (11)$$

where $K = R^{-1}B^T S$ and S is the solution to the ARE:

$$0 = A^T S + SA - SBR^{-1}B^T S + Q \quad (12)$$

Replacing the values for $\bar{u}$ and $\bar{x}$ in (11):

$$\begin{aligned} u(t) &= -Kx(t) + KA^{-1}(CA^{-1}B)^{-1}r - (CA^{-1}B)^{-1}r \\ &= -Kx(t) + \left[KA^{-1}B - I\right]\left[CA^{-1}B\right]^{-1} r \\ &= -Kx(t) + Fr \end{aligned} \quad (13)$$

where F is given by:

$$F = \left[KA^{-1}B - I\right]\left[CA^{-1}B\right]^{-1} \quad (14)$$

Assuming that all states x are measurable, a block diagram of this control strategy is given below.

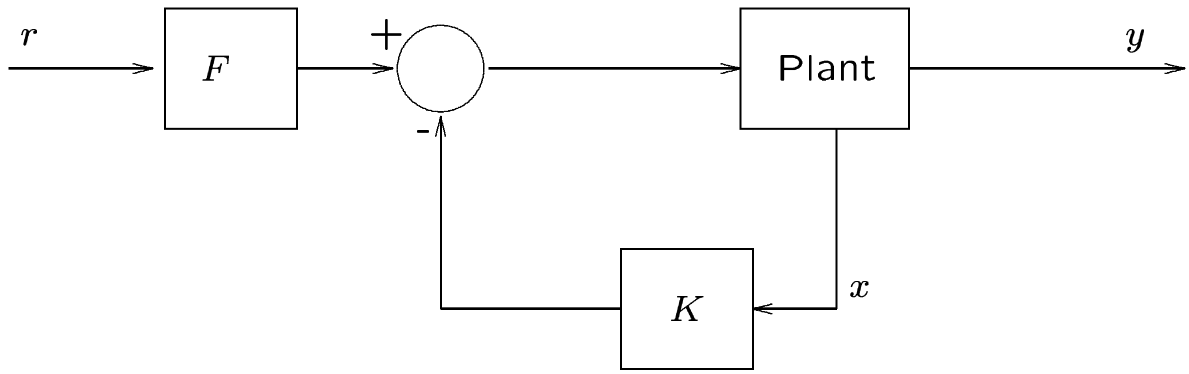

Figure 1: Block diagram of LQ tracking controller

Provided the above assumptions are satisfied, this LQ tracking controller provides $y = r$ in the steady state, as the deviations $x - \bar{x}$ and $u - \bar{u}$ are regulated about zero.

However, in the presence of step disturbances affecting either the outputs or the states, there will be a steady state error.

Output error weighting

Note that if the performance index has the form:

$$J = \frac{1}{2}\int_0^\infty \left\{(y-r)^T Q_y (y-r) + \tilde{u}^T R \tilde{u}\right\} dt$$

it can be reduced to the state error weighting form:

$$J = \frac{1}{2}\int_0^\infty \left\{\tilde{x}^T Q \tilde{x} + \tilde{u}^T R \tilde{u}\right\} dt$$

by using $Q = C^T Q_y C$ (Homework: prove it).

Example

Consider the system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and a performance index:

$$J = \frac{1}{2}\int_0^{\infty} \left\{ (y-r)^T Q_y (y-r) + \tilde{u}^T R \tilde{u} \right\} dt$$

where

$$Q_y = \text{diag}(10, 1), \quad R = \text{diag}(0.3, 0.3)$$

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% Matlab code using the Control Systems Toolbox
A = [0 1 0;
     0 0 1;
     3 1 -3]
B = [1 0
     0 0
     0 1];
C = [1  0  0
     0  1  0];
D = zeros(2,2);
Qy = diag([10 1]);
R = diag([0.3, 0.3]);
[n n] = size(A);
[ny n] = size(C);
Q= C'*Qy*C
[K S] = lqr(A,B,Q,R);
F = (K/A*B-eye(ny))/(C/A*B);
```

The resulting values for K and F are:

$$K = \left[\begin{array}{ccc} 6.0004 & 2.3001 & 0.4844 \\ 0.4844 & 2.9087 & 0.8187 \end{array} \right]$$

$$F = \left[\begin{array}{cc} 6.0004 & 1.3001 \\ -2.5156 & 1.9087 \end{array} \right]$$

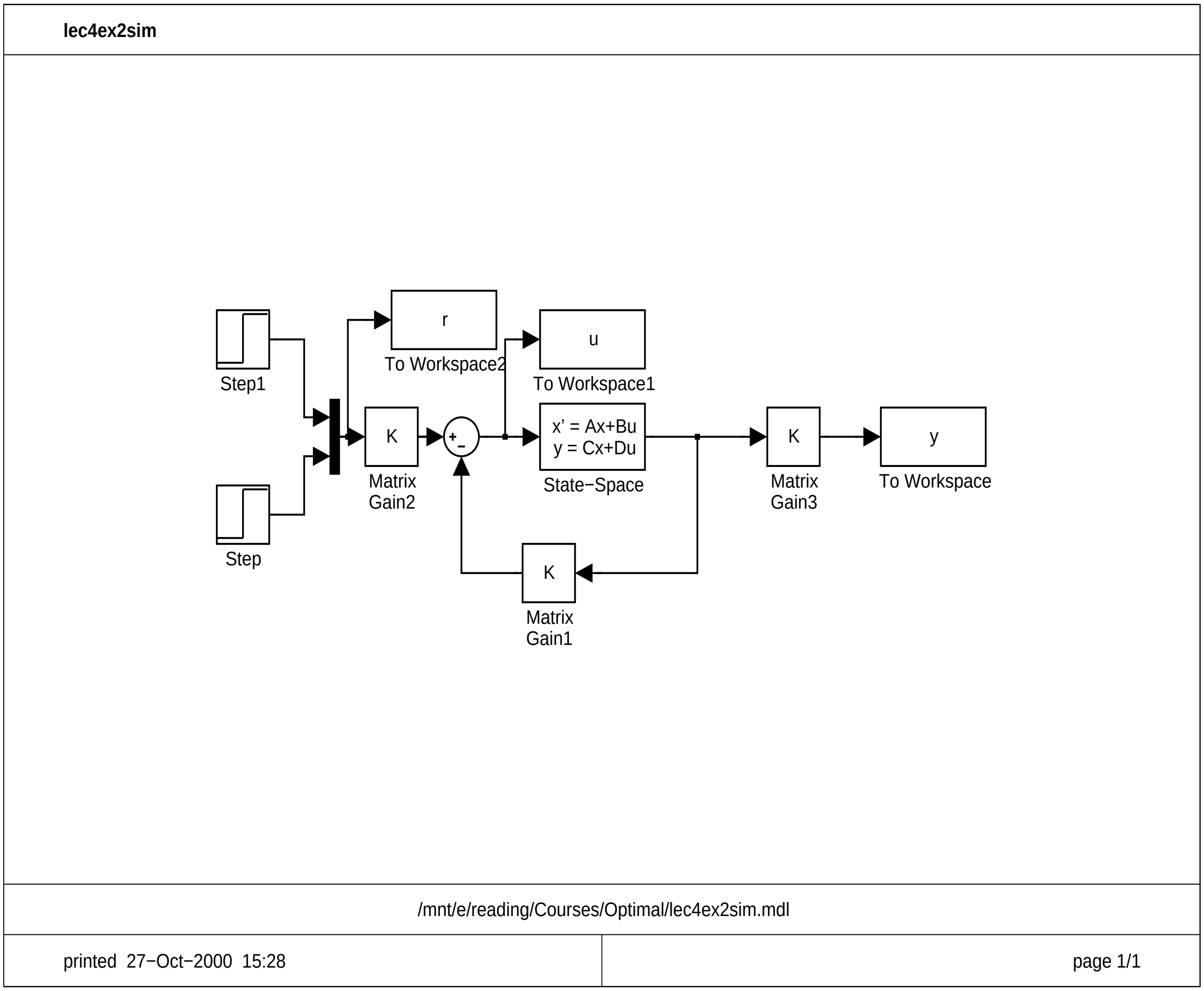

Figure 2: Simulink block diagram.

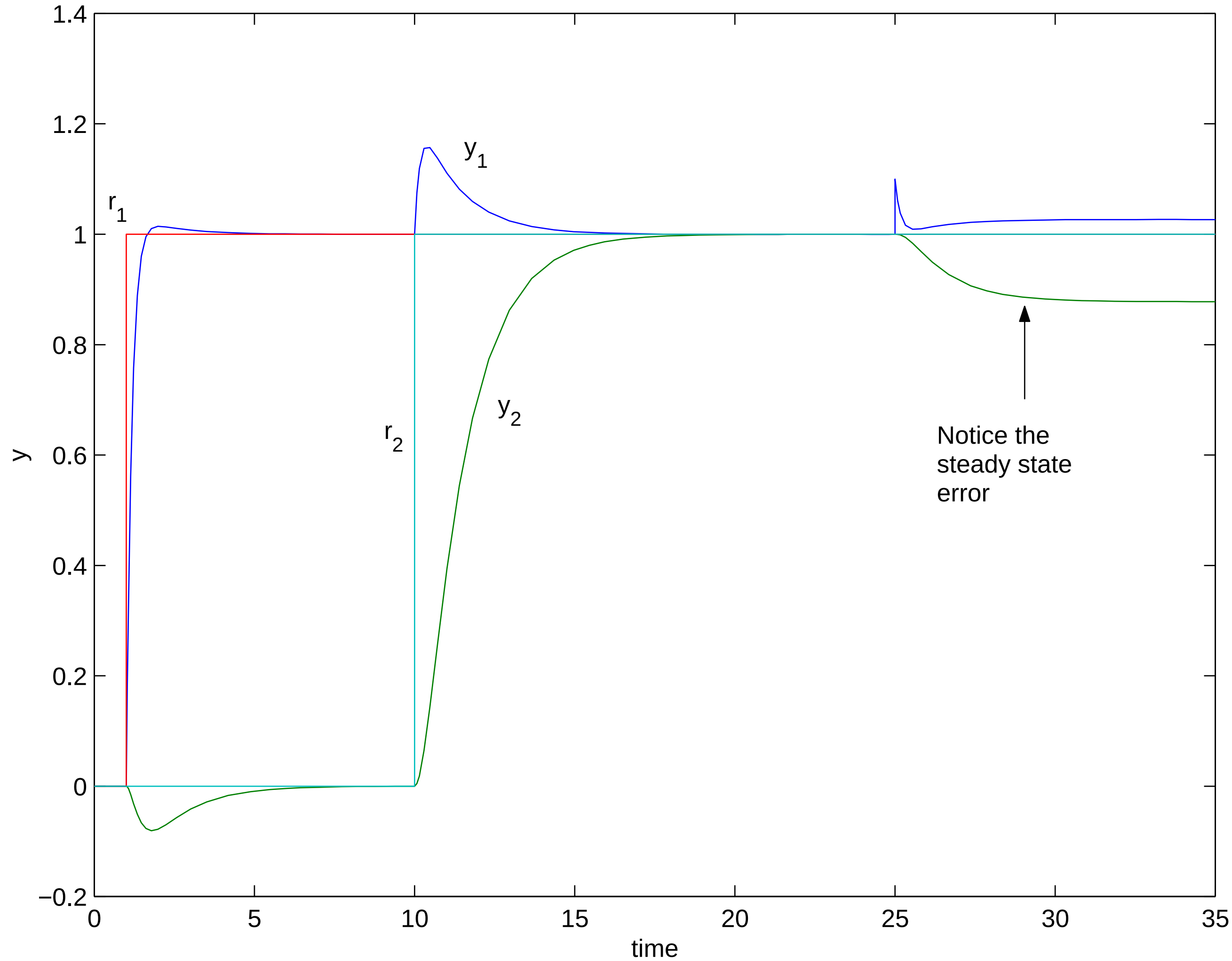

Figure 3: Simulation results after step changes in r, showing the effect of a state step disturbance with value $d = [0.1,\ 0,\ 0]^T$.