Tracking optimal control with integral action

Integral action is used in classical control to eliminate steady state errors when tracking constant signals.

Integral control can be generated in a LQ setting by considering the integral of the output error as an extra set of state variables..

The integral of the tracking error is generated by the following differential equation:

$$\dot{w} = r - y(t) = r - Cx(t) \qquad (1)$$

Define an augmented state error $\widehat{x} = \begin{bmatrix} \tilde{x} \\ w \end{bmatrix}$

Then the augmented error state equation is:

$$\dot{\hat{x}}(t) = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{\hat{A}} \hat{x} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\hat{B}} \tilde{u} \qquad (2)$$

A suitable performance index would be:

$$J = \int_0^\infty \hat{x}^T Q \hat{x} + \tilde{u}^T R \tilde{u}, \;\; Q \geq 0, R > 0 \qquad (3)$$

The optimal control law is as follows:

$$\tilde{u} = -K\hat{x} \qquad (4)$$

where

$$K = R^{-1}\hat{B}^T S = [K_x \;\; K_w] \qquad (5)$$

S is the solution to the ARE:

$$0 = \hat{A}^T S + S\hat{A} - S\hat{B}R^{-1}\hat{B}^T S + Q \qquad (6)$$

using the definitions for $\hat{x}$ and $\tilde{u}$, we have:

$$u = -K_x\tilde{x} - K_w w + \bar{u} \tag{7}$$

but $\tilde{x} = x - \bar{x}$, therefore:

$$u = -K_x(x - \bar{x}) - K_w w + \bar{u} \tag{8}$$

we have seen before that $\bar{u} = -[CA^{-1}B]^{-1}r$ and $\bar{x} = A^{-1}B[CA^{-1}B]^{-1}r$. The control law is then:

$$u = -K_x x - K_w w + K_r r \tag{9}$$

where:

$$K_r = [K_x A^{-1}B - I][CA^{-1}B]^{-1} \tag{10}$$

A block diagram of this LQI tracking controller is as follows

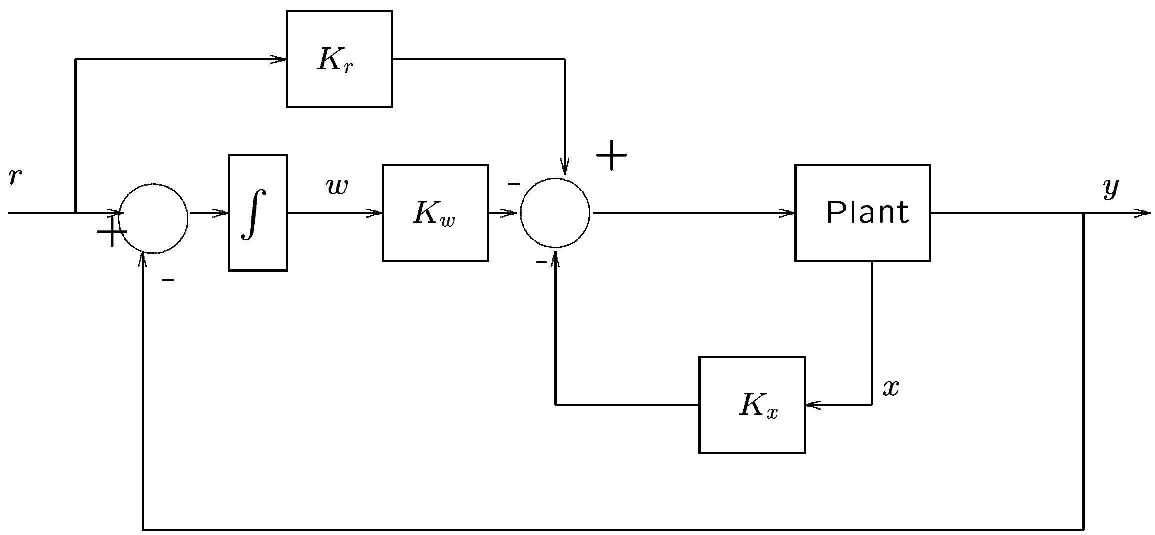

Figure 1: Block diagram of the LQI tracking controller

Output error weighting

Note that if the performance index has the form:

$$J = \frac{1}{2}\int_0^\infty \left\{(y-r)^T Q_y (y-r) + w^T Q_w w + \tilde{u}^T R \tilde{u}\right\} dt$$

it can be reduced to the state error weighting form:

$$J = \frac{1}{2}\int_0^\infty \left\{\hat{x}^T Q \hat{x} + \tilde{u}^T R \tilde{u}\right\} dt$$

by using

$$Q = \begin{bmatrix} C^T Q_y C & 0 \\ 0 & Q_w \end{bmatrix} \qquad (11)$$

Example

Consider the system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and a performance index:

$$J = \frac{1}{2} \int_0^\infty \left\{ (y-r)^T Q_y (y-r) + w^T Q_w w + \tilde{u}^T R \tilde{u} \right\} dt$$

where

$$Q_y = Q_w = \mathrm{diag}(10, 1), \quad R = \mathrm{diag}(0.3, 0.3)$$

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% Matlab code using the Control Systems Toolbox
A = [0 1 0;     0 0 1;     3 1 -3]
B = [1 0;     0 0;     0 1];
C = [1  0  0;     0  1  0];
ny = 2; nu = 2; nx = 3;
Ahat = [A , zeros(nx,ny)
        -C, zeros(ny,ny)];
Bhat = [ B
         zeros(ny,nu)];
Chat = [C zeros(ny,ny)];
Qy = diag([10 1]);
Qw = diag([10 1]);
R = diag([0.3, 0.3]);
Qx = C'*Qy*C;
Q = [Qx          , zeros(nx,ny)
     zeros(ny,nx) ,      Qw     ];
[K S] = lqr(Ahat,Bhat,Q,R);
Kx = K(1:nu,1:nx);
Kw = K(1:nu,nx+1:nx+nu);
Kr = (Kx/A*B-eye(ny))/(C/A*B);
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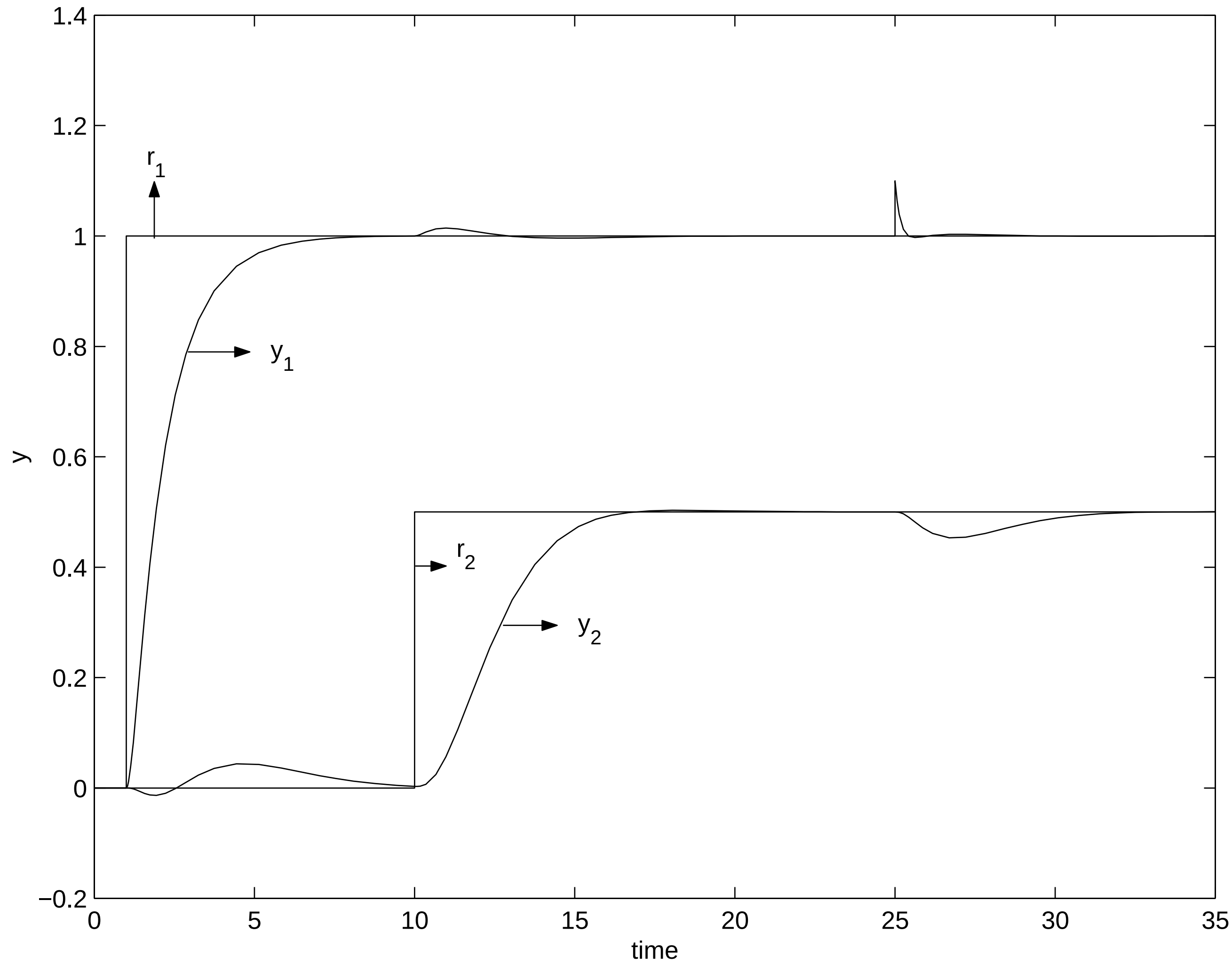

Figure 2: Simulation results after step changes in r, showing the effect of a state step disturbance with value $d = [0.1,\ 0,\ 0]^T$ entering at 25 s.

LQG Optimal Control

LQG stands for linear quadratic Gaussian. It is a way of implementing the linear quadratic regulator when not all states are measured. A Kalman filter is used to provide state estimates.

LQG is an important application of optimal control theory.

Introduction to the Kalman filter

The Kalman filter is a stochastic filter that provides optimal state estimates for a system described by a linear state space model that is subject to Gaussian noise.

Asume that a system is described by the following discrete time state space model:

$$x_{k+1} = Ax_k + Bu_k + Gw_k \tag{12}$$

where w_k is a p-dimensional sequence of uncorrelated random variables with zero mean and covariance matrix Q_f, which represents uncertainties in the model.

Assume the following measurement model:

$$y_k = Cx_k + v_k \tag{13}$$

where v_k is a r–dimensional sequence of uncorrelated random variables with zero mean and covariance matrix R_f, which represents measurement noise.

The steady state Kalman filter is computed as follows. First, find the state error covariance matrix P by solving the following algebraic Riccati equation.

$$P = A[P - PC^T(CPC^T + R_f)^{-1}CP]A^T + GQ_fG^T \tag{14}$$

Second, compute the Kalman filter gain from the following equation:

$$K_f = PC^T \left[CPC^T + R_f\right]^{-1} \tag{15}$$

The discrete Kalman filter involves the following recursive computations, which are initialized with a guess for the state estimate $\widehat{x}(0) = \bar{x}_0$.

1. Given the previous state estimate $\widehat{x}_{k-1}$ and the latest control input u_{k-1}, define the a priori estate estimate $\widehat{x}_k^-$ as follows:

$$\widehat{x}_k^- = A\widehat{x}_{k-1} + Bu_{k-1} \tag{16}$$

2. Given the a priori estimate $\widehat{x}_k^-$, the Kalman gain K_f and the current measurement $y_{m,k}$, the current state estimate $\widehat{x}_k$ is found as follows:

$$\widehat{x}_k = \widehat{x}_k^- + K_f \left[y_{m,k} - C\widehat{x}_k^- \right] \tag{17}$$

Uniqueness and stability

Three conditions must be satisfied for a unique solution of P to exist and the steady state Kalman filter to be stable.

- First, R_f must be positive definite.
- Second, the pair $(A, G\sqrt{Q_f})$ must be reachable.
- Third, the pair (A, C) must be detectable.

Definition 1 (Reachability) *The pair* (A, B) *is reachable if the eigenvalues of* $(A - BK)$ *can be arbitrarily assigned by appropriate choice of the feedback matrix* K*.*

Definition 2 (Detectability) *The pair (A,C) is detectable if* $(A - CL)$ *can be made asymptotically stable by some matrix* L*.*

LQG Control - discrete case

Given the Kalman filter formulated above, it can be coupled with the discrete time LQ regulator. This is called an LQG controller.

Recall the discrete LQ regulator equations:

Riccati equation:

$$S = A^T \left[S - SB(B^T SB + R)^{-1} B^T S \right] A + Q \tag{18}$$

State feedback gain matrix:

$$K = \left[B^T SB + R \right]^{-1} B^T SA \tag{19}$$

The optimal control is:

$$u_k = -Kx_k \tag{20}$$

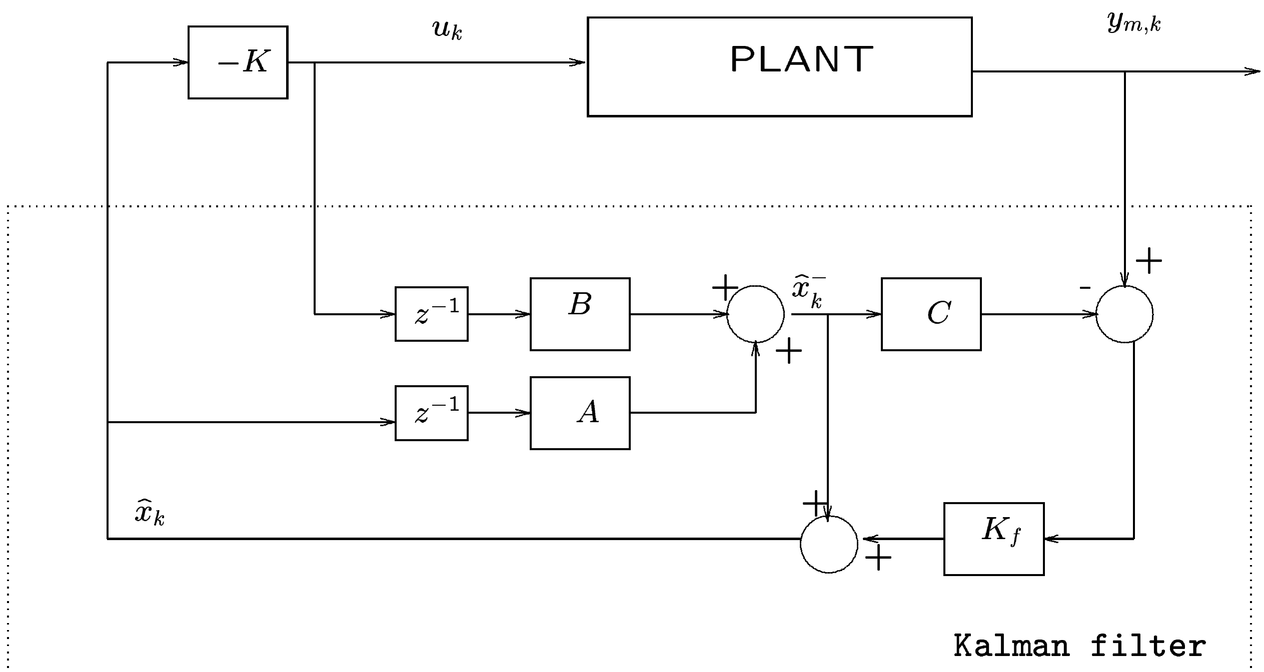

Figure 3: Discrete LQG control

LQG control – continuous case

The rationale for the continuous case is similar to that given for the discrete case. A summary of the relevant equations follows:

System model:

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t) \tag{21}$$

$$y(t) = Cx(t) + v(t) \tag{22}$$

It is assummed that $w(t)$ and $v(t)$ are uncorrelated random processes with covariance matrices Q_f and R_f, respectively.

Initialization:

$$\widehat{x}(0) = \bar{x}_0 \tag{23}$$

Error covariance algebraic Riccati equation:

$$AP + PA^T + GQ_fG^T - PC^TR_f^{-1}CP = 0 \quad (24)$$

Kalman filter gain:

$$K_f = PC^TR_f^{-1} \quad (25)$$

State estimate dynamics:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_f\left[y_m(t) - C\hat{x}(t)\right] \quad (26)$$

LQ regulator Algebraic Riccati Equation:

$$0 = A^T S + SA - SBR^{-1}B^T S + Q \qquad (27)$$

LQ regulator gain

$$K = R^{-1}B^T S \qquad (28)$$

LQG control law

$$u(t) = -K\hat{x}(t) \qquad (29)$$

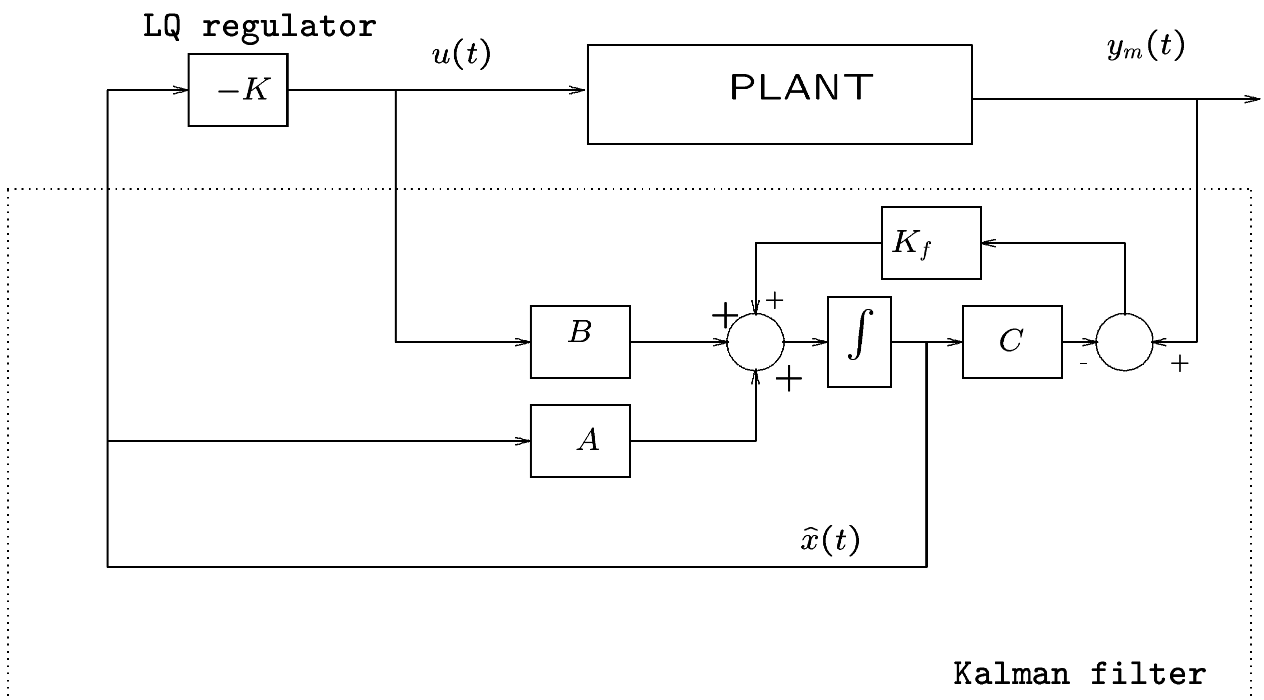

Figure 4: Continuous LQG control

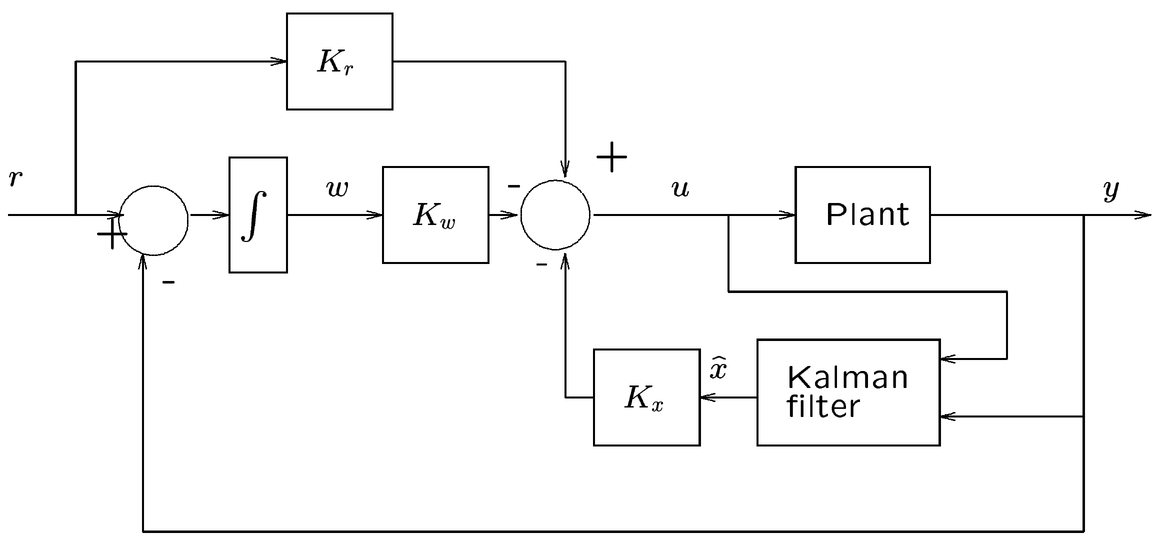

Figure 5: Continuous LQG tracking controller with integral action

Example

Consider the system with noisy measured output:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx + v(t) \end{aligned}$$

where $v(t)$ has zero mean value white noise with covariance R_f,

$$R_f = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and a performance index:

$$J = \frac{1}{2}\int_0^\infty \left\{ (y-r)^T Q_y (y-r) + w^T Q_w w + \tilde{u}^T R \tilde{u} \right\} dt$$

where

$$Q_y = Q_w = \mathrm{diag}(10, 1), \quad R = \mathrm{diag}(0.3, 0.3)$$

The Kalman filter uses:

$$Q_f = \mathrm{diag}(0.001, 0.001, 0.001)$$

```
% Design of the Kalman filter using Matlab
% and the Control Systems Toolbox command lqe

G = eye(3);

Qf = 0.001*eye(3);

Rf = 0.001*eye(2);

[Kf Pf] = lqe(A,G,C,Qf,Rf);
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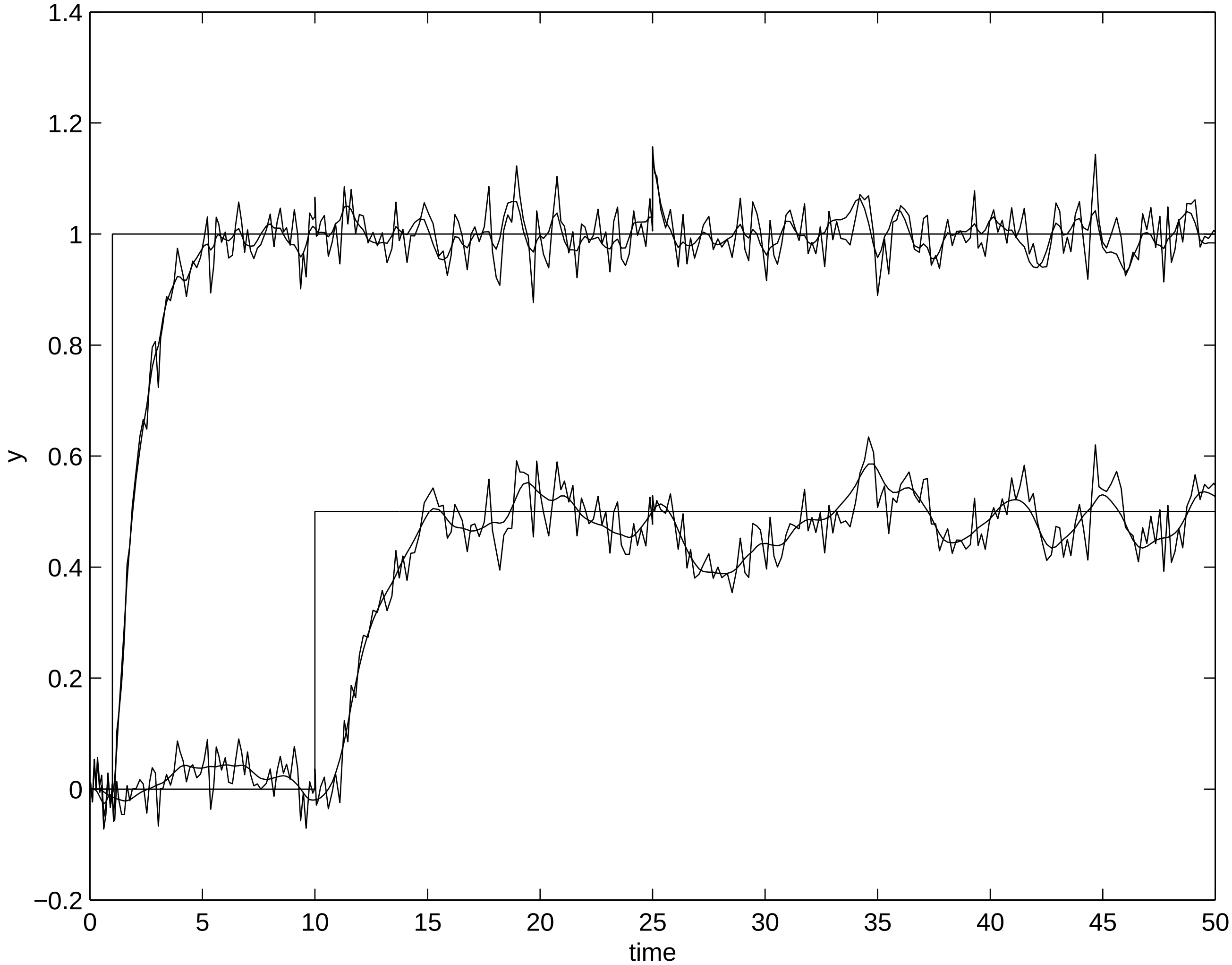

Figure 6: LQG tracker simulation results after step changes in r, showing the effect of a state step disturbance with value $d = [0.1,\ 0,\ 0]^T$ entering at 25 s.