CHAPTER 2:
REVIEW OF FREQUENCY DOMAIN ANALYSIS

The long-term response of a process is known as the frequency response which is obtained from the response of a complex-domain transfer function. The frequency response analysis is useful for:

- Calculating the input-output dynamic characteristics.
- Analyzing dynamic systems and designing controllers.

2.1 Shortcut method for Frequency response

Step 1: For a given s-domain transfer function ($G(s)$), set $s = j\omega$ to get $G(j\omega)$

Step 2: Express $G(j\omega)$ in terms of $R + jI$ using complex conjugate multiplication.

Step 3: compute the amplitude ratio as $|G(j\omega)| \equiv AR = \sqrt{R^2 + I^2}$ and the phase angle as $\prec G(j\omega) \equiv \phi = \tan^{-1}(I/R)$

2.2 properties of Frequency response

For a complex transfer function such as:

$$G(s) = \frac{G_a(s)G_b(s)G_c(s)\cdots}{G_1(s)G_2(s)G_3(s)\cdots} \qquad 1$$

The amplitude ratio is simply:

$$|G(j\omega)| = \frac{|G_a(j\omega)|\,|G_b(j\omega)|\,|G_c(j\omega)|\cdots}{|G_1(j\omega)|\,|G_2(j\omega)|\,|G_3(j\omega)|\cdots} \qquad 2$$

The phase angle is simply:

$$\prec G(j\omega) = \phi_a + \phi_b + \phi_c + \cdots - \phi_1 - \phi_2 - \phi_3 \qquad 3$$

2.3 Bode Plot

Bode diagram is used to display $G(j\omega)$ where AR and ϕ are plotted as a function of frequency ω. These plots are useful for:

- Rapid analysis of the response characteristic
- Analyzing stability of closed-loop systems.

2.4 Examples

2.4.1 First Order System

Let the process model be:

$$p(s) = \frac{k}{\tau s + 1} \tag{4}$$

$$p(j\omega) = \frac{k}{\tau j\omega + 1} \tag{5}$$

$$p(j\omega) = \frac{k(-\tau j\omega + 1)}{(\tau j\omega + 1)(-\tau j\omega + 1)} = \frac{k(-\tau j\omega + 1)}{\tau^2\omega^2 + 1} =$$

$$\frac{k}{\tau^2\omega^2 + 1} + j\frac{k(-\tau\omega)}{\tau^2\omega^2 + 1} = R + jI \tag{6}$$

Therefore;

$$R(j\omega) = \frac{k}{\tau^2\omega^2 + 1} \tag{7}$$

$$I(j\omega) = \frac{-k(j\omega)}{\tau^2\omega^2 + 1} \tag{8}$$

Consequently:

$$AR = |G(j\omega)| = \sqrt{\left(\frac{k}{\tau^2\omega^2 + 1}\right)^2 + \left(\frac{-k\tau\omega}{\tau^2\omega^2 + 1}\right)^2} =$$

$$k\sqrt{\frac{1 + \tau^2\omega^2}{\left(\tau^2\omega^2 + 1\right)^2}} = \frac{k}{\sqrt{\left(\tau^2\omega^2 + 1\right)}} \tag{9}$$

$$\prec G(j\omega) = \phi = \tan^{-1}(-\omega\tau) = -\tan^{-1}(\omega\tau) \tag{10}$$

Sketch and Analysis of the frequency response:

At low frequency, i.e. $\omega \rightarrow 0$ ($\omega << 1/\tau$):

$$\text{AR} = k$$

$$\phi = 0$$

At high frequency, i.e. $\omega \rightarrow \infty$ ($\omega >> 1/\tau$):

$$AR = 1/\omega\tau$$

$$\phi = -90^{o}$$

At the break point (corner frequency), i.e. ω = 1/τ:

$$AR = \frac{k}{\sqrt{1+1}} = \frac{k}{\sqrt{2}} = 0.707; \qquad for\ \ k = 1$$

$$\phi = \tan^{-1}(-1) = -45^{o}$$

For quick sketch of the frequency response compute the slope of the amplitude ratio at high frequency as:

$$\log AR = \log 1 - \log \omega\tau = -\log \omega\tau$$

Then plot the frequency response on a log-log scale as follows:

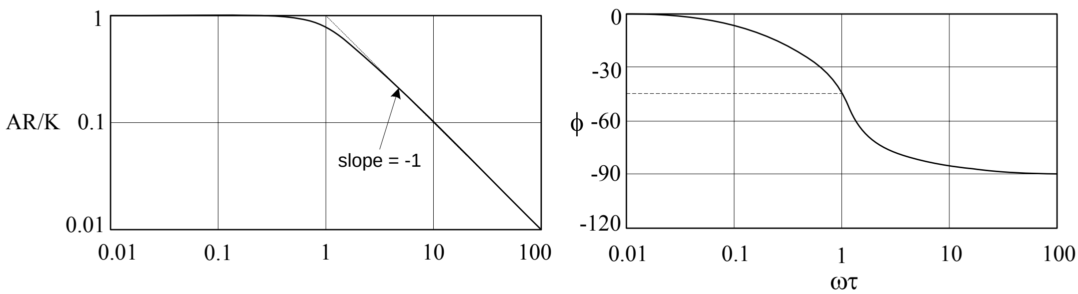

Figure 1: Frequency response for a first-order system

2.4.2 Second Order System

A general second order system is given by:

$$p(s) = \frac{k}{\tau^2 s + 2\zeta s + 1} \qquad 11$$

Following the above technique we can show that:

$$AR = \frac{k}{\sqrt{(1-\omega^2\tau^2)^2 + (2\omega\tau\zeta)^2}} \qquad 12$$

$$\phi = \tan^{-1}\left[\frac{-2\omega\tau\zeta}{1-\omega^2\tau^2}\right] \qquad 13$$

For over-damped system, $\zeta > 1$

At low frequency:

$$AR_N = AR/K = 1$$

$$\phi = -\tan^{-1}(0) = 0$$

At high frequency:

$$AR_N = AR/K \approx \frac{1}{\omega^2\tau^2}$$

$$\phi = -\tan^{-1}(0) = -180^o$$

At corner frequency:

$$AR_N = AR/K = \frac{1}{\sqrt{2}} = 0.707$$

$$\phi = -\tan^{-1}(\infty) = -90^o$$

The slope at high frequency:

$$\log AR = \log 1 - 2\log \omega\tau = -2\log \omega\tau$$

The sketch is almost similar to that of a first order system:

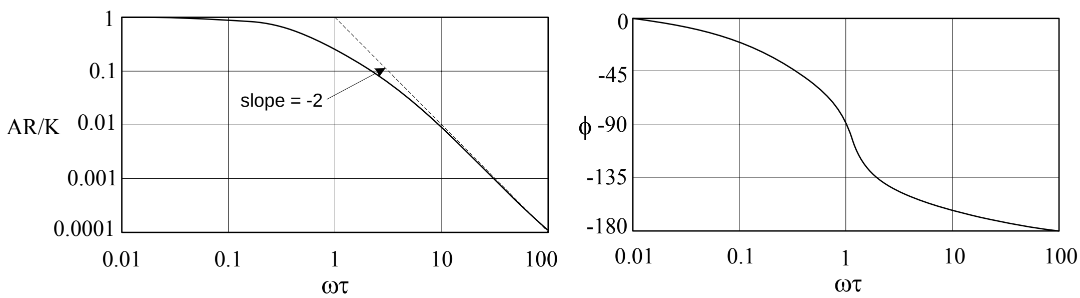

Figure 2: Frequency response for second-order system

For under-damped system, the bode plot is almost the same except that it may have a maximum amplitude ratio. Taking the derivative of equation 12 with respect to ω, gives:

$$\frac{d}{d\omega}AR = \frac{2\omega\tau^2(1-\tau^2\omega^2) - (2\tau\zeta)^2\omega}{\left[(1-\tau^2\omega^2)^2 + (2\tau\omega\zeta)^2\right]^{3/2}} \qquad 14$$

Setting the last equation to zero gives:

$$\omega = \frac{\sqrt{1-2\zeta^2}}{\tau} \qquad 15$$

Substituting the value of ω into equation 12 yields:

$$AR_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad 16$$

Thus, the Bode plot looks like:

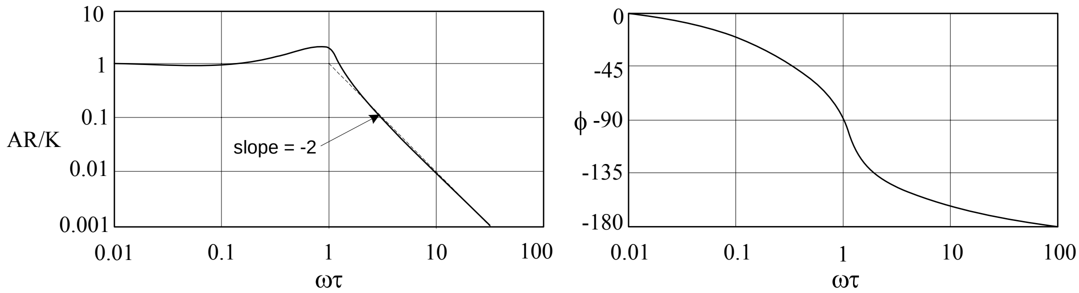

Figure 3: Frequency response for under-damped system

The frequency at which the maximum occur (equation 15) is known as the resonant frequency because at that frequency the sinusoidal output response has the maximum amplitude for a given sinusoidal input.

2.4.3 Time delay

$$p(s) = e^{-\theta s} \qquad 17$$

$$p(j\omega) = e^{-\theta jw} = \cos\omega\theta - j\sin\omega\theta \qquad 18$$

$$AR = \sqrt{\cos^2\omega\theta + \sin^2\omega\theta} = 1 \qquad 19$$

$$\phi = \tan^{-1}\left(-\frac{\cos\omega\theta}{\sin\omega\theta}\right) = \omega\theta \qquad 20$$

Therefore, the sketch of this frequency system is:

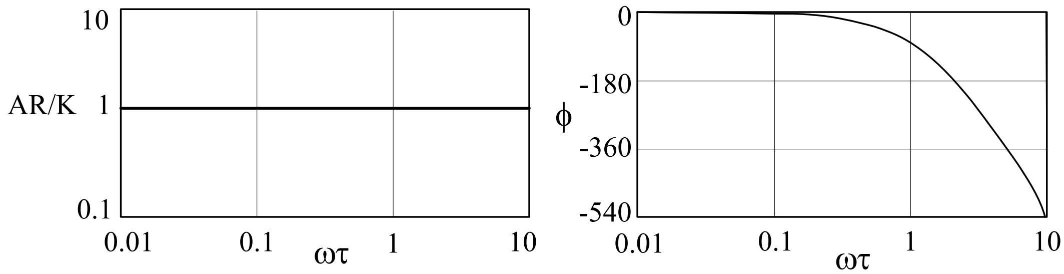

Figure 4: Frequency response for a time delay

2.4.4 Zero (Lead) processes

$$p(s) = \tau s + 1 \tag{21}$$

$$p(j\omega) = \tau j w + 1 = 1 + j\tau\omega = R + jI \tag{22}$$

$$AR = |p(j\omega)| = \sqrt{1^2 + (\omega\tau)^2} \tag{23}$$

$$\phi = \tan^{-1}(\omega\tau) \tag{24}$$

At low frequencies, $\omega \rightarrow 0$

$$AR = 1$$
$$\phi = \tan^{-1}(0) = 0$$

At intermediate frequency, $\omega = 1/\tau$

$$AR = \sqrt{2}$$
$$\phi = \tan^{-1}(1) = 45^o$$

At high frequencies, $\omega >> 1$

$$AR = \omega\tau$$
$$\phi = \tan^{-1}(\infty) = 90^0$$

The slope: $\log AR = \log \omega\tau$

The quick sketch of the frequency response is:

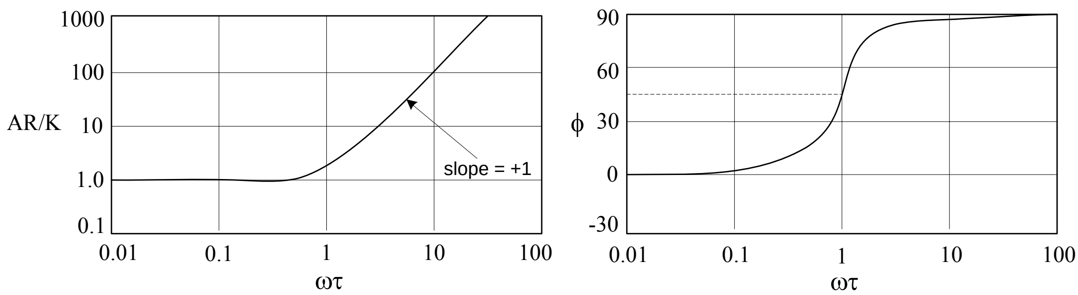

Figure 5: Frequency response of a lead process

Comments

Lag processes:

- A lag processes always have a negative phase angle which indicates that the output follows or lag the input by ϕ.
- For a lag processes, the amplitude ratio always approaches zero for high frequencies.

Lead processes (zero processes)

- The amplitude ratio becomes very large at high frequencies.
- The phase angle is positive except in the presence of RHP zeros.
- Processes with RHP zeros or time delay are known as nonminimum phase systems because they exhibit phase lag.

2.4.5 Composite system

$$G(s) = \frac{5(0.5s+1)e^{-0.5s}}{(20s+1)(4s+1)} \qquad 21$$

According to the properties in equations (1) and (2) we have:

$$AR = |G(j\omega)| = \frac{|5|\,|0.5j\omega+1|\,|e^{-0.5j\omega}|}{|20j\omega+1|\,|4j\omega+1|} = \frac{(5)\left(\sqrt{(0.5\omega)^2+1}\right)(1)}{\left(\sqrt{(20\omega)^2+1}\right)\left(\sqrt{(4\omega)^2+1}\right)} \equiv \frac{|G_1||G_2||G_3|}{|G_4||G_5|} \qquad 22$$

$$\phi = -\tan^{-1}(20\omega) - \tan^{-1}(4\omega) + \tan^{-1}(0) + \tan^{-1}(0.5\omega) - 0.5\omega \qquad 23$$

$$\log AR = \log(5) + \frac{1}{2}\left(\log(0.5\omega)^2+1\right) + \log(1) - \frac{1}{2}\left(\log(20\omega)^2+1\right) - \frac{1}{2}\left(\log(4\omega)^2+1\right) \qquad 24$$

For quick sketch the bode plot we have:

For the amplitude ratio, simply accumulate the slopes at high frequencies descending from the largest time constant:

$$\log AR = -\log 20\omega - \log 4\omega + \log 0.5\omega \qquad 25$$

This means the slope should be:

Frequency	slope
$0 < \omega < 1/20$	0
$1/20 < \omega < 1/4$	–1
$1/4 < \omega < 1/0.5$	–2
$1/0.5 < \omega < \infty$	+1

For the phase angle simply sum all phase angles as shown in Figure 6.

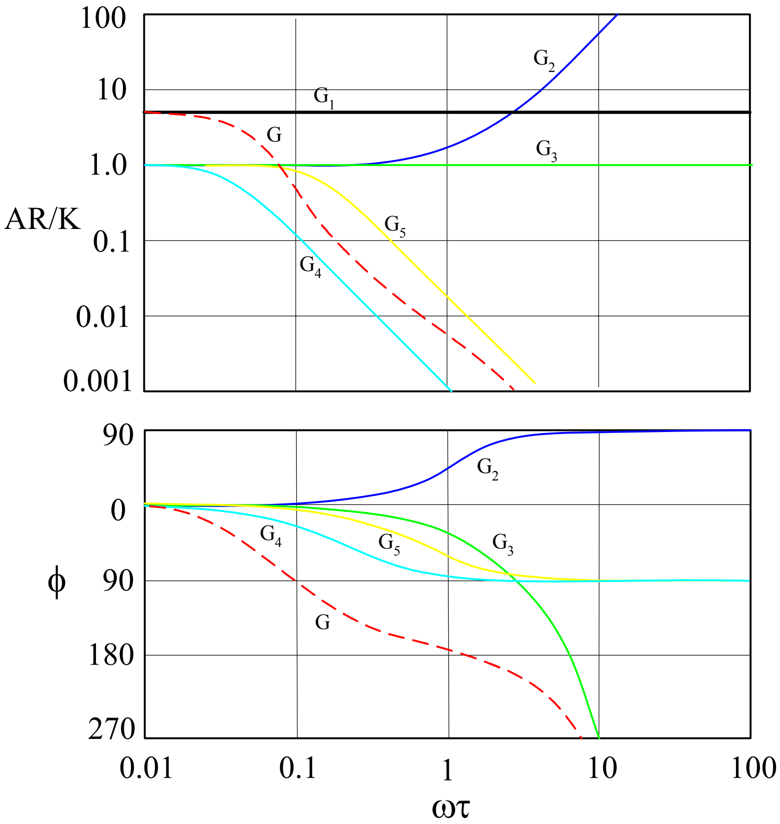

Figure 6: Frequency response for composite system

2.5 Bode Stability Criteria

A closed-loop system is unstable if the frequency response of the open-loop transfer function $G_{ol} = G_c G_v G_p G_m$ has an amplitude ratio greater than one at the critical frequency. Otherwise the close-loop system is stable.

The critical frequency ω_c is defined as to be the frequency at which the open-loop phase angle is -180°.

Limitations:
Bode stability criteria can not be used for unstable process, which have multiple critical frequencies.

For these types of processes, *Nyquist stability* criteria might be used.

Example 2.1: Given

$$G_p(s) = \frac{2}{(0.5s+1)^3}; \qquad G_c(s) = k_c; \qquad G_v(s) = 0.`; \qquad G_m(s) = 10$$

$$G_{OL}(s) = \frac{2k_c}{(0.5s+1)^3}$$

The bode plot for this system at three different values for k_c is shown o in Figure 7.

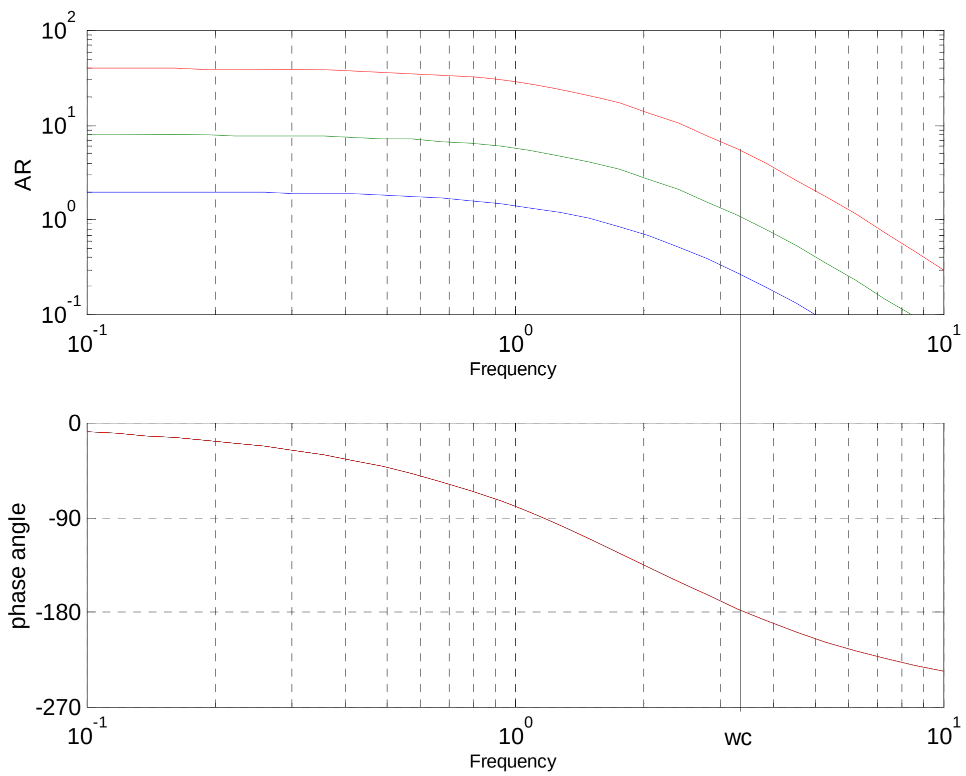

Figure 7: Bode diagram for three values for k_c.

It obvious that:

At $k_c = 1$ the system is stable because AR << 1 at $\omega = \omega_c$

At $k_c = 4$ the system is marginally stable because AR = 1 at $\omega = \omega_c$

At $k_c = 20$ the system is unstable because AR >> 1 at $\omega = \omega_c$

2.6 Effect of controller on frequency response

A typical frequency response for PID controller:

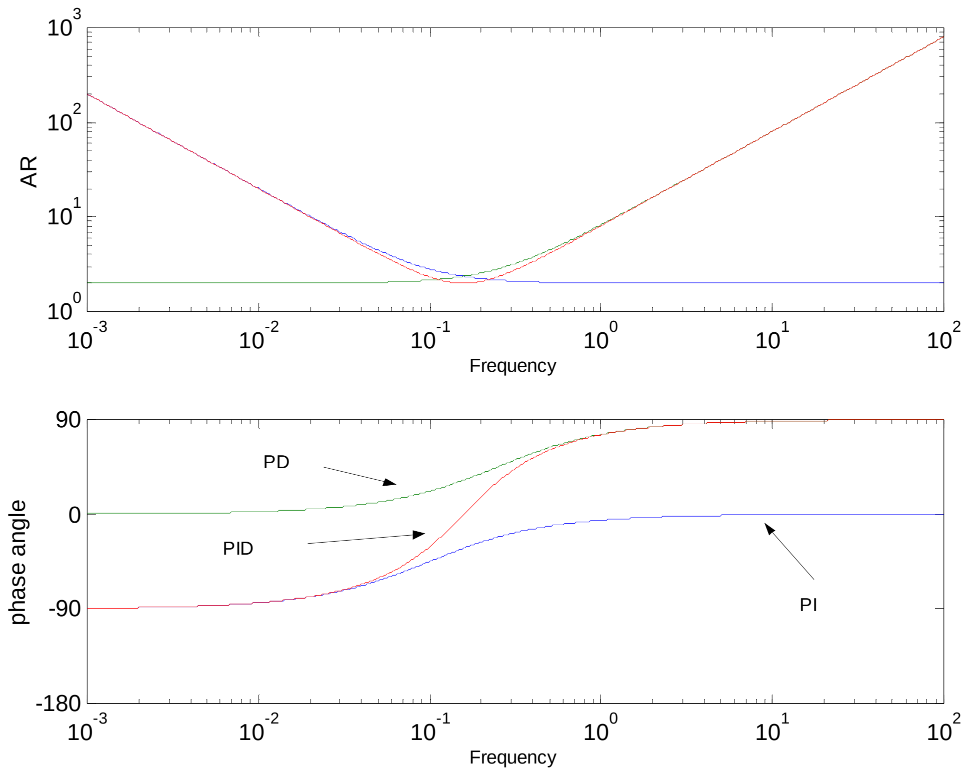

Figure 8: Typical frequency response for three modes of PID controller

- Integral action is included in controller to eliminate offset. However, it adds *phase lag* making the system less stable. In this case, the phase angle curve decreases rapidly, thus, the chances for the phase lag of high-order system with PI controller to cross -180° at low frequencies is higher.

- Derivative action adds *phase lead* improving stability and allowing higher gains to be used to improve the closed-loop response.

Tuning is beneficial to achieve:

- A large value for ω_c is also desirable since it indicates small close-loop response time.
- It only desirable that the amplitude ratio be small at ω_c but it can be increased at other frequencies to improve control system performance.

Example 2.2

$$G_p(s) = \frac{5}{(s+1)(0.5s+1)}; \; G_c(s) = k_c; \qquad G_v(s) = 1 \qquad G_m(s) = 1$$

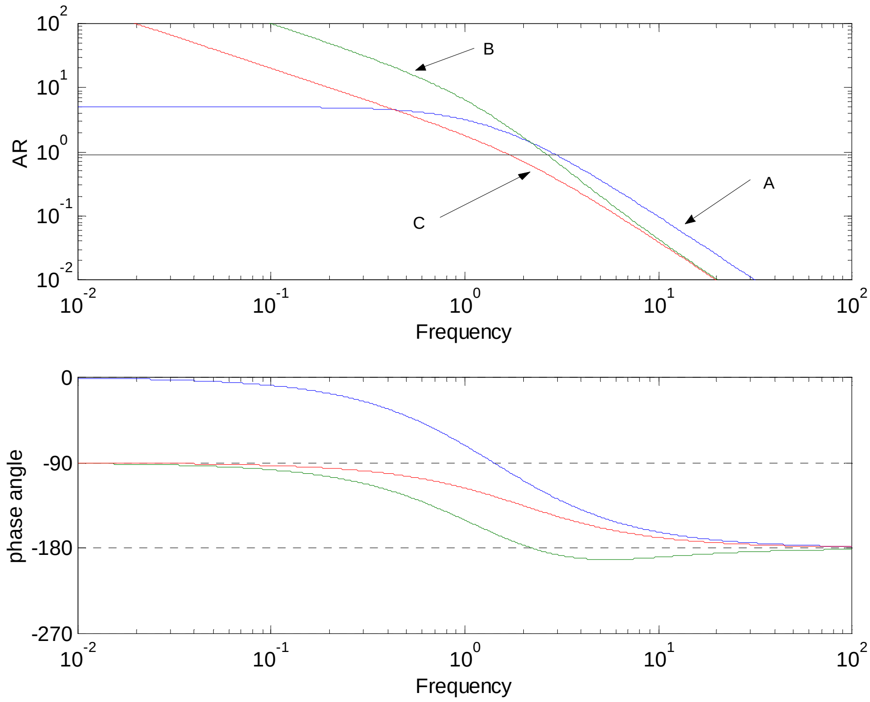

Figure 9: Bode Diagram for Example 5.2; curve A: proportional controller; curve B: PI controller with k_c = 0.4, τ_I = 0.2; curve C: PI controller with k_c = 0.4, τ_I = 1

- It is clear from curve A in Figure 9, that a proportional control does not add any phase lag. Note that the critical frequency does not exist because $\phi > -180^o$ for all frequencies. Hence, k_c can be extremely large and the closed-loop system will always be stable. k_{cu} is infinity in this case.

- The inclusion of integral action in the controller can cause the closed-loop system to become unstable. Curve B in Figure 9 shows that the phase angle crosses -180^{o} with AR >>1 for G_c = 0.4(1+1/0.2s).

- Curve C indicates that when τ_I is increased to 1, a stable closed-loop response results for all values of k_c because there is no critical frequency.

2.7 Gain and Phase Margin

As a measure or relative stability, the term gain and phase margins are used. The gain margin is defined as:

$$GM = \frac{1}{AR_c} \tag{26}$$

Here AR_c is the value of the open-loop amplitude ratio at the critical frequency. Since AR_c must be less than one for stability, then GM > 1 is a stability requirement.

If ω_g is the frequency at which the open-loop gain (AR) is unity and ϕ_g is the phase angle at that frequency, then:

$$PM = 180 + \phi_g \tag{27}$$

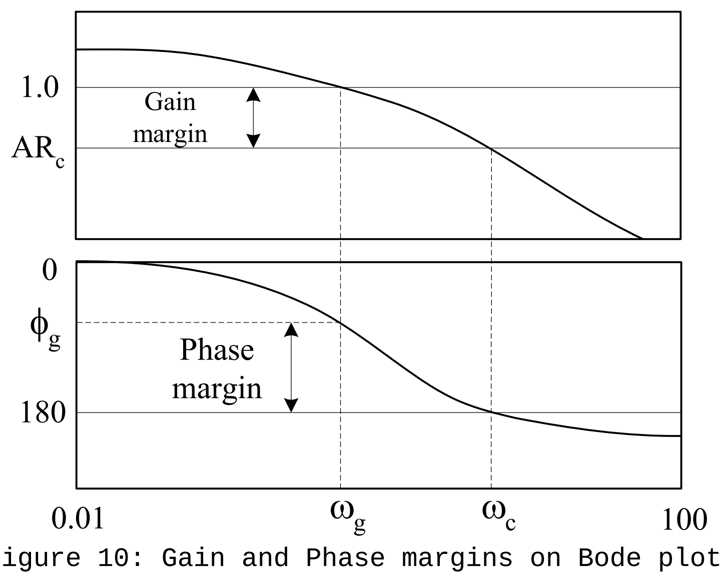

Figure 10: Gain and Phase margins on Bode plot

Remarks:

- Controller manufacturers recommend that a well-tuned controller has a gain margin of 1.7 to 2 and a phase margin of 30 to 45^{o}.
- GM and PM can be used to provide a comparison between good performance and stability.
- Large values for GM and PM cause sluggish close-loop response, while small values result in a less sluggish, more oscillatory response.
- The concept of gain and phase margin does not apply for processes with multiple critical frequencies.

2.8 Closed-loop Frequency domain

The amplitude ratio and phase angle for closed-loop response is given by:

$$M = \frac{|y(j\omega)|}{|r(j\omega)|} = \left| \frac{G_v G_p G_c}{1 + G_v G_p G_m G_c} \right| \quad 28$$

$$\psi = \prec \frac{y(j\omega)}{r(j\omega)} = \prec \frac{G_v G_p G_c}{1 + G_v G_p G_m G_c} \quad 29$$

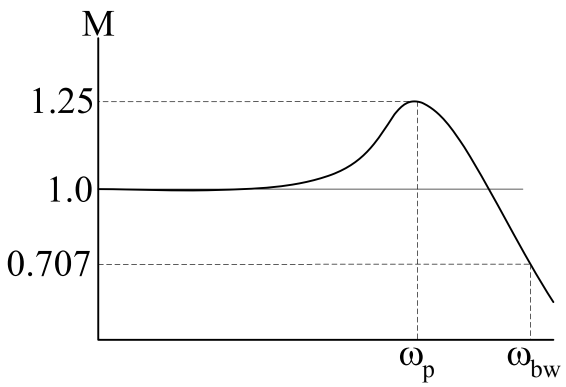

Figure 11: Closed-loop response

Comments:

- **M** should be unity as ω →0 for set point and zero as ω→ 0 for disturbance indicating no offset.
- For set point, M should be maintained as unity up to as high frequency as possible, while for disturbance, M should be minimized over as wide a frequency range as possible. This is to ensure a rapid approach to steady state.

- The peak amplitude ratio at the resonant frequency should no larger than 1.25 corresponding to a damping ratio of $\zeta = 0.5$
- The controller should be tuned such that ω_p is as large as possible. A large value for ω_p implies faster response to set point.
- The bandwidth ω_{bw} is the frequency at which $M = \sqrt{2}/2 = 0.707$. A large value for ω_{bw} indicates a relatively fast response with a short rise time.

ω_b	break (corner) frequency	$\omega = 1/\tau$
ω_c	critical frequency	$-180 = -\tan^{-1}(\omega_c)$
ω_p	resonance frequency	$M_{max}(\omega_p)$